ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 6, 2015

LIOUVILLIAN FIRST INTEGRALS FOR QUADRATIC SYSTEMS WITH AN INTEGRABLE SADDLE

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ABSTRACT. We provide explicit expressions for the Liouvillian first integrals of the quadratic polynomial differential systems having an integrable saddle.

1. Introduction. Let $\mathbb{R}[x,y]$ be the ring of all polynomials in the variables x and y and with coefficients in $\mathbb{R}$.

A *quadratic polynomial differential system* or simply a *quadratic system* is a polynomial differential system in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$ and the maximum of the degrees of P and Q is 2.

Quadratic differential systems have been widely studied in the last 100 years, and more than 1,000 papers have been published about them (see, for instance, [**12, 16, 17**]). These systems are considered as one of the easiest, but not trivial, families of nonlinear differential systems, although the problem of classifying all quadratic vector fields (even integrable ones) still remains open. For more information on the integrable differential vector fields in dimension 2, see for instance, [**3**]).

The classification of the centers for the quadratic systems has a long history which started with the works of Dulac [**5**], Kapteyn [**9, 10**], Bautin [**2**], Zoladek [**18**], etc. Schlomiuk, Guckenheimer and Rand in [**13,** pages 3, 4 and 13] described a brief history of the problem of the

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2010 AMS *Mathematics subject classification.* Primary 34A34, 34C05, 34C14.

Keywords and phrases. quadratic systems, integrable saddle, Liouvillian first integral, integrating factor, inverse integrating factor.

The second author is supported by grants MICINN/FEDER MTM 2008-03437, Generalitat de Catalunya 2009SGR410, ICREA Academia and FP7-PEOPLE-2012-IRSES-316338. The third author is supported by grants AGAUR PIV-DGR-2010 and by the FCT through the project PTDC/MAT/ 117106/2010 and through CAMGD, Lisbon.

Received by the editors on December 29, 2012, and in revised form on January 21, 2014.

DOI:10.1216/RMJ-2015-45-6-1765