STABILITY OF SINGULAR LIMIT CYCLES FOR ABEL EQUATIONS

J.L. BRAVO, M. FERNÁNDEZ, A. GASULL

Abstract. We obtain a criterion for determining the stability of singular limit cycles of Abel equations $x' = A(t)x^3 + B(t)x^2$. This stability controls the possible saddle-node bifurcations of limit cycles. Therefore, studying the Hopf-like bifurcations at $x = 0$, together with the bifurcations at infinity of a suitable compactification of the equations, we obtain upper bounds of their number of limit cycles. As an illustration of this approach, we prove that the family $x' = at(t-t_A)x^3 + b(t-t_B)x^2$, with $a, b > 0$, has at most two positive limit cycles for any t_B, t_A.

1. Introduction and Main results

The study of the number of periodic solutions of Abel differential equations is a challenging question. These equations are interesting because they provide models of real phenomena, see for instance [4, 9, 12], or as a tool for studying several subcases of Hilbert XVI problem on the number of limit cycles of planar polynomial differential equations, see [7, 14].

In this paper we consider Abel equations,

$$\frac{dx}{dt} = x' = A(t)x^3 + B(t)x^2, \tag{1.1}$$

with $A(t), B(t)$ continuous functions defined on $[0, T]$. Let $u(t, x)$ denote the solution of (1.1) determined by $u(0, x) = x$. We say $u(t, x)$ is *closed* or *periodic*, if $u(T, x) = x$, and *singular* or *multiple*, if it is closed and $u_x(T, x) = 1$. When $u_x(T, x) \neq 1$ then it is said that it is *simple* or *hyperbolic*. Isolated closed solutions are also called *limit cycles* and a singular closed solution such that $u_{xx}(T, x) \neq 0$ will be called a *double closed solution*, or also a *semistable limit cycle*.

Notice that $x = 0$ is always a closed solution of (1.1). Therefore the number of limit cycles in regions $x > 0$ and $x < 0$ can be studied separately. Since one region can be sent to the other one with the transformation $x \to -x$, we will restrict our attention to the region $x > 0$.

There are several results for uniqueness of limit cycles of (1.1) on $x > 0$. The most known ones impose that one of the functions A or B does not change sign, see [10, 11, 14, 17, 19]. Other conditions, allowing A and B

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changing sign, are given for instance in [1, 5]. In this paper we consider simple Abel equations for which there is no uniqueness of positive limit cycles and study their number by controlling the nature of the double closed solutions.

As a motivating example, consider the Abel equation

$$x' = at(t-t_A)x^3 + b(t-t_B)x^2, \quad a,b \in \mathbb{R}^+,\, t_B, t_A \in \mathbb{R} \tag{1.2}$$

with $T=1$. In any of the following cases, the known methods allow to prove that (1.2) has at most one (simple) positive limit cycle:

(1) $t_A \notin (0,1)$ or $t_B \notin (0,1)$.
(2) $t_A \le t_B$.

In case (1) it is a consequence of the result stated above and appearing in [10, 11, 14, 17, 19], because either A or B have no zeroes in $(0,1)$. In case (2), it is proved in [1] that if for some $\alpha, \beta \in \mathbb{R}$ the function $\alpha A + \beta B$ does not vanish identically and does not change sign in $(0,1)$ then the Abel equation has at most one positive limit cycle. Hence, if we consider $\alpha = 1/a$, $\beta = -1/b$ we get that for every $t \in [0,1]$,

$$\alpha A(t) + \beta B(t) = t(t-t_A) - (t-t_B) > (t_B - t_A) \ge 0.$$

and the result follows in this case. As a consequence of our main results, Theorems 1.2 and 1.3 below, we prove:

Theorem 1.1. *Abel equation* (1.2) *has at most two positive limit cycles, taking into account their multiplicities, and this upper bound is sharp.*

As we will see, the existence of two positive limit cycles is because for $t_B = 1/2$ and $t_A = 2/3$ the multiplicity of the closed solution $x=0$ is four, while generically it is two. Hence a Hopf-like codimension two bifurcation appears and two positive limit cycles can be created from this solution.

Before stating our general results we also describe the global bifurcation diagram of the positive limit cycles of (1.2) when $a=b=1$ in the plane $(t_B, t_A) \in [0,1]^2$, see Figure 1. On each region R_j, the integer value indicates the number of positive limit cycles. The curves T_∞ and T_d correspond to bifurcations at infinity of some limit cycle and to the existence of a double limit cycle, respectively. The line $t_B = 1/2$ corresponds to a Hopf-like bifurcation at $x=0$ of codimension one, except at the point $(1/2, 2/3)$, which corresponds to a Hopf-like bifurcation of codimension two. We have fixed the values $a=b=1$ to simplify the explanation, but the techniques that we introduce to produce this diagram can also be applied to the other cases giving similar results.

The curves T_∞ and T_d of Figure 1 are qualitative rather than exact. They are obtained using essentially two different tools: the first one is that the functions A and B depend monotonically with respect to the parameters t_A and t_B, respectively, see the definition below; and the second one is the use of a compactification of the differential equation, see Section 3.2. This compactication has also been used in [15, 16] for studying second kind Abel

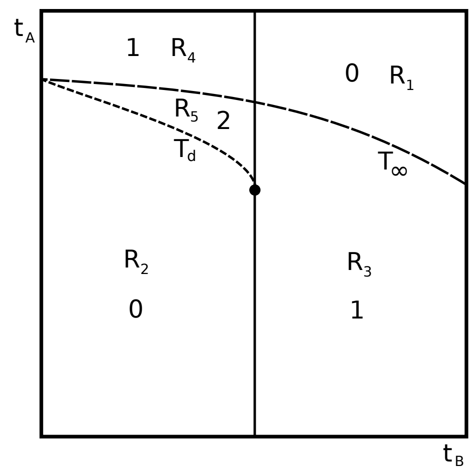

FIGURE 1. Bifurcation diagram of the positive limit cycles of (1.2) with $a = b = 1$ for $(t_B, t_A) \in [0, 1]^2$.

and generalized Abel equations. In particular, it allows to introduce a new critical point of the equation "at infinity" that turns out to be a hyperbolic saddle. As we will see, the control of its separatrices will be the key point to know the behavior of the curve T_∞. Moreover, studying the stability of a certain new type of "polycycles" we derive a tool for obtaining the stability of infinity, that we believe that will be useful to get a full understanding of other families of Abel equations, see Proposition 3.3 and Remark 3.4.

Our main result for the general Abel equation (1.1) is the following:

Theorem 1.2. *Consider Abel equation* (1.1) *and assume that:*

- (C_1) *A, B have a unique simple zero in $(0, T)$ denoted by t_A, t_B, respectively. Moreover, $t_A \neq t_B$.*
- (C_2) *For each positive singular closed solution u and each $\gamma \in \mathbb{R}$, the function $u(t, x) - \gamma\phi(t)$ has at most two simple zeroes, where*

$$\phi(t) = -\frac{B(t)}{2A(t)}.$$

Moreover, if $0 < t_1 < t_2 < T$ are simple zeroes of $u - \gamma\phi$, then $t_1 < t_A < t_2$.

Then $\operatorname{sgn} u_{xx}(T, x) = \operatorname{sgn}\Big((t_B - t_A)B(t_A)A(t_B)\Big)$.

A difficult point for applying the above result is to verify when hypothesis (C_2) holds. Notice that this hypothesis includes the unknown singular closed solution. Nevertheless, in Proposition 3.1 we give a sufficient algebraic condition, so that it can be checked computationally.

When both functions A and B depend on a parameter λ in a certain manner our main result can be rewritten in a more suitable way. We will say that (1.1) is *monotonic with respect to* λ at λ_0 if either

$$A_\lambda(t,\lambda_0) \geq 0, \quad B_\lambda(t,\lambda_0) \geq 0, \text{ for every } t \in [0,T],$$

or

$$A_\lambda(t,\lambda_0) \leq 0, \quad B_\lambda(t,\lambda_0) \leq 0, \text{ for every } t \in [0,T],$$

and $A^2_\lambda(t,\lambda_0) + B^2_\lambda(t,\lambda_0) \not\equiv 0$, where in this paper, the subindexes in a function indicate the corresponding partial derivatives.

Theorem 1.3. *Assume that Abel equation* (1.1) *is monotonic with respect to* λ *for* $\lambda \in [\lambda_1, \lambda_2]$*, and* $(C_1), (C_2)$ *hold for every* $\lambda \in (\lambda_1, \lambda_2)$*. Let* $N(\lambda_0)$ *denote the number of positive limit cycles of* (1.1) *for* $\lambda = \lambda_0$*, taking into account their multiplicities. Assume that* $N(\lambda_1)$ *and* $N(\lambda_2)$ *are finite and the positive limit cycles for* $\lambda = \lambda_1, \lambda_2$ *are simple. Then, for* $\lambda \in (\lambda_1, \lambda_2)$*,*

$$N(\lambda) \leq \max\big(N(\lambda_1), N(\lambda_2)\big) + 2.$$

Moreover, if no closed solution bifurcate from the origin when $\lambda \in [\lambda_1, \lambda_2]$*, then*

$$N(\lambda) \leq \max\big(N(\lambda_1), N(\lambda_2)\big) + 1.$$

Notice that the above definition of monotonic with respect to λ for families of Abel equations is an adaptation to this setting of the so called *rotated families of planar vector fields* introduced by Duff in 1953, see [8] or [18, Sec. 4.6]. For these families of vector fields, the control of semistable bifurcations of limit cycles is also crucial for understanding their global bifurcation diagram of limit cycles.

2. Proofs of the main results

2.1. **Proof of Theorem 1.2.** We shall divide the proof of Theorem 1.2 into several Propositions. In the following we assume that $u(t,x)$ is a singular positive solution, and that $(C_1), (C_2)$ hold.

Proposition 2.1. *For any* $\alpha, \beta \in \mathbb{R}$*,*

$$\operatorname{sgn} u_{xx}(T,x) = \operatorname{sgn}\left(\int_0^T F(t,\alpha)G(t,\beta)\,dt\right), \tag{2.3}$$

where

$$F(t,\alpha) := (2-\alpha)B(t) + 2(3-\alpha)A(t)u(t,x),$$
$$G(t,\beta) := u_x(t,x) - \beta u(t,x).$$

Proof. Deriving in (1.1) with respect to x and using again (1.1), we have for any $\alpha \in \mathbb{R}$,

$$u_x(t,x) = \exp\left(\int_0^t (2-\alpha)Bu(t,x) + (3-\alpha)Au^2(t,x) + \alpha\frac{u'}{u}\,dt\right). \tag{2.4}$$

Evaluating at $t = T$, we have

(2.5)
$$u_x(T,x) = \left(\frac{u(T,x)}{x}\right)^{\alpha} \exp\left(\int_0^T (2-\alpha)Bu(t,x) + (3-\alpha)Au^2(t,x)\,dt\right).$$

Since $u(T,x) = x$ and $u_x(T,x) = 1$, evaluating expression above at $\alpha = 2, 3$, we obtain

$$\int_0^T B(t)u(t,x)\,dt = \int_0^T A(t)u^2(t,x)\,dt = 0. \tag{2.6}$$

Deriving with respect to x the formula (2.5) and taking into account (2.6), we obtain that the sign of $u_{xx}(T,x)$ coincides with the sign of

$$\begin{aligned}
&\int_0^T \left((2-\alpha)B(t) + 2(3-\alpha)A(t)u(t,x)\right) u_x(t,x)\,dt \\
&= \int_0^T \left((2-\alpha)B(t) + 2(3-\alpha)A(t)u(t,x)\right)\left(u_x(t,x) - \beta u(t,x)\right) dt \\
&= \int_0^T F(t,\alpha)G(t,\beta)\,dt.
\end{aligned}$$

□

With equation (2.3) in mind, the idea for proving Theorem 1.2 is to choose α and β such that the corresponding $F(t,\alpha)$ and $G(t,\beta)$ share their zeroes, which are simple, and as a consequence $F(t,\alpha)G(t,\beta)$ does not change sign. This is the goal of next three propositions.

Proposition 2.2. *For every α, $F(t,\alpha)$ has at most two simple zeroes in $(0,T)$. Moreover*

1. *$F(t,\alpha) = 0$ is the graph of a continuous function $\alpha(t)$ defined for every t such that $u(t,x) \neq \phi(t)$.*
2. *For every $\bar{t}$ such that $u(\bar{t},x) = \phi(\bar{t})$,*
$$\lim_{t\to\bar{t}} |\alpha(t)| = \infty,$$
and $\operatorname{sgn}\alpha(t) = \operatorname{sgn}(u(t,x) - \phi(t))$ *for t close to $\bar{t}$.*
3. *$\alpha(t)$ is monotonic on every interval in its domain of definition.*

Proof. By (C_1), $F(t,2) = 2A(t)u(t,x)$ and $F(t,3) = -B(t)$ have at most one simple zero in $[0,T)$.

Let $\alpha \neq 2,3$. Note that $F(t_A,\alpha) \neq 0$. If $A(t) \neq 0$, then

$$F(t,\alpha) = 2A(t)\big((3-\alpha)u(t,x) - (2-\alpha)\phi(t)\big).$$

Thus $F(t,\alpha) = 0$ if and only if $u(t,x) = \gamma\phi(t)$, with $\gamma = \frac{2-\alpha}{3-\alpha}$. By hypothesis (C_2), $F(t,\alpha)$ has at most two simple zeroes.

Moreover, $F(t,\alpha)=0$ if and only if $\alpha=\alpha(t)$, where

$$\alpha(t)=\begin{cases}2+\dfrac{u(t,x)}{u(t,x)-\phi(t)}, & \text{if } t\neq t_A,\\ 2, & \text{if } t=t_A,\end{cases}$$

is continuous in every $t\in[0,T]$ such that $u(t,x)-\phi(t)\neq 0$. So (1) and (2) hold.

For $\alpha=2$, $F(t,2)=2A(t)u(t,x)$ has a zero at $t=t_A$, thus, $\alpha(t_A)=2$. Now, there exists at most one simple zero of $F(t,\alpha)$ for $t\in(0,t_A)$ and at most one for $t\in(t_A,T)$. Since $\alpha(t)$ is continuous, then it is monotonic in both intervals. □

Proposition 2.3. *There exist β_0,β_1,β_2, such that $G(t,\beta)$ has two simple zeroes in $(0,T)$ for every $\beta\in(\beta_1,\beta_2)$, $\beta\neq\beta_0$, and no zeroes for $\beta\notin[\beta_1,\beta_2]$. Moreover,*

(1) *$G(t,\beta)=0$ is the graph of a closed continuous function $\beta(t)$ defined for $t\in[0,T]$.*
(2) *$G(t,\beta)>0$ for $\beta<\beta(t)$ and $G(t,\beta)<0$ for $\beta>\beta(t)$.*
(3) *$\beta'(t)=0$ if and only if $u(t,x)=\phi(t)$.*
(4) *If $u(t,x)-\phi(t)\neq 0$ for every t in an interval I and $t_A\in I$, then $\operatorname{sgn}\beta'(t)=\operatorname{sgn}B(t_A)$.*

Proof. The function

$$\beta(t)=\frac{u_x(t,x)}{u(t,x)}$$

satisfies (1). Since $G_\beta(t,\beta)=-u(t,x)$,(2) also holds.

Since $u(t,x),u_x(t,x)>0$ and

$$\beta'(t)=\big(2A(t)u^2(t,x)+B(t)u(t,x)\big)\,\beta(t),$$

to prove (3) it is sufficient to prove that $2A(t)u^2(t,x)+B(t)u(t,x)=0$ if and only if $u(t,x)=\phi(t)$.

If $A(t)=0$, then $B(t)\neq 0$ since A,B do not have common zeroes. If $A(t)\neq 0$, then $2A(t)u^2(t,x)+B(t)u(t,x)=0$ is zero if and only if $u(t,x)=\phi(t)$.

By (3), for any interval I such that $u(t)-\phi(t)\neq 0$, for every $t\in I$, $\operatorname{sgn}\beta'(t)=\operatorname{sgn}G_t(t,\beta)$ does not change. To obtain (4), note that

$$G_t(t_A,\beta(t_A))=B(t_A)u(t_A)u_x(t_A).$$

Then

$$\beta'(t_A)=-\frac{G_t(t_a,\beta(t_A))}{G_\beta(t_A,\beta(t_A))}=B(t_A)u_x(t_A).$$

Since $u(t,x)$ is singular, $u(t,x),u_x(t,x)$ are closed, then $\beta(0)=\beta(T)=:\beta_0$. By hypothesis (C_2), $u(t,x)-\phi(t)$ has at most two simple zeroes. Therefore, $\beta(t)$ has at most one maximum and at most one minimum. From that, in $(0,T)$ there are exactly two solutions of $\beta(t)=\beta$ for every $\beta\in(\min(\beta(t)),\max(\beta(t)))$, $\beta\neq\beta_0$, and the proof concludes. □

Proposition 2.4. *There exist α, β such that $F(t,\alpha)$ and $G(t,\beta)$ have common zeroes. Let $I = \int_0^T F(t,\alpha)G(t,\beta)\,dt$. Then, for $t_A \neq t_B$,*

$$\operatorname{sgn} I = \operatorname{sgn}\big((t_B - t_A)B(t_A)A(t_B)\big).$$

Proof. By (C_2), $u(t,x) - \phi(t)$ hast at most two simple zeroes in $(0,T)$. Since $\beta(t)$ is continuous and periodic, then it has an extremum in $(0,T)$. By Proposition 2.3, $\beta'(t) = 0$ if and only if t is a zero of $u(t,x) - \phi(t)$. Then $u(t,x) - \phi(t)$ has at least one zero in $(0,T)$.

Assume that the only zero of $u(t,x) - \phi(t)$ in $(0,T)$ is t_1. Since $\beta(t)$ has a unique extremum at t_1, and it is strictly monotonic for $t \neq t_1$, then there exist two continuous monotonic functions T_1, T_2, defined in (β_1, β_2) such that $\beta \circ T_k = \mathrm{Id}$, $T_1(\beta) < t_1 < T_2(\beta)$, and T_1, T_2 have opposite monotonicity. Define the continuous function

$$d(\beta) = \alpha(T_1(\beta)) - \alpha(T_2(\beta)), \quad \beta \in (\beta_1, \beta_2).$$

Since α is monotonic and T_1, T_2 have opposite monotonicity, then d is monotonic. We have

$$\{\alpha(T_1(\beta))\colon \beta \in (\beta_1,\beta_2)\} = \begin{cases} (\alpha(0), +\infty) & \text{if } \alpha \text{ is increasing,} \\ (-\infty, \alpha(0)) & \text{if } \alpha \text{ is decreasing,} \end{cases}$$

$$\{\alpha(T_2(\beta))\colon \beta \in (\beta_1,\beta_2)\} = \begin{cases} (-\infty, \alpha(T)) & \text{if } \alpha \text{ is increasing,} \\ (\alpha(T), +\infty) & \text{if } \alpha \text{ is decreasing.} \end{cases}$$

Moreover, if $d(\beta) \neq 0$ for every $\beta \in (\beta_1, \beta_2)$, then there exists α such that $\alpha(t) \neq \alpha$, thus, $F(t,\alpha)$ has no zeroes. Taking

$$\beta > \sup\{\beta(t)\colon t \in [0,T]\},$$

$G(t,\beta) > 0$ for every $t \in [0,T]$ and therefore I has a given sign. But taking the same α and $\beta < \inf\{\beta(t)\colon t \in [0,T]\}$, then I has exactly the opposite sign. From this contradiction, there exists no singular solution in this case. Therefore, there exists β such that $d(\beta) = 0$. Let $\alpha = \alpha(T_1(\beta)) = \alpha(T_2(\beta))$. Then $T_1(\beta) < T_2(\beta)$ are the common changes of sign of $F(t,\alpha)$ and $G(t,\beta)$.

To establish $\operatorname{sgn} I$, assume that $0 < t_A < t_1 < t_B < T$. (The rest of the cases are similarly handled).

For α, β such that F, G have common changes of sign, $F(t,\alpha)$ must have exactly two changes of sign (we recall $G(t,\beta)$ has two changes of sign). Since $\alpha(t)$ is a monotonic function, the sign of $F(t,\alpha)$ between the two changes of sign is the sign of $F(t_B, 2) = 2A(t_B)u(t_B, x)$.

From $\operatorname{sgn} \beta'(t) = \operatorname{sgn} B(t_A)$ for $t < t_1$, we have that β is increasing for $t < t_1$, when $B(t_A) > 0$. Therefore t_1 is a maximum of β and $G(t,\beta) > 0$ in $(T_1(\beta), T_2(\beta))$ by Proposition 2.3 (2). Analogously, β is decreasing for $t < t_1$, when $B(t_A) < 0$, the point t_1 is a minimum of β and $G(t,\beta) < 0$ in $(T_1(\beta), T_2(\beta))$. Therefore, the sign of $G(t,\beta)$ between the two changes of sign is the sign of $B(t_A)$.

In conclusion, the sign of I is the sign of $B(t_A)A(t_B)$.

Assume that $0 < t_1 < t_2 < T$ are the changes of sign of $u(t,x) - \phi(t)$. Then,

$$\lim_{t \to t_k \pm} \alpha(t) = \pm\infty \quad \text{or} \quad \lim_{t \to t_k \pm} \alpha(t) = \mp\infty,$$

for $k = 1, 2$. Then the image of (t_1, t_2) by $\alpha(t)$ is $\mathbb{R}$. In particular, as $B(t)$ has only one change of sign in $(0,T)$, $t_B \in (t_1, t_2)$. Moreover, 3 is not in the image by $\alpha(t)$ of $(0, t_1) \cup (t_2, T)$, see Figure 2.

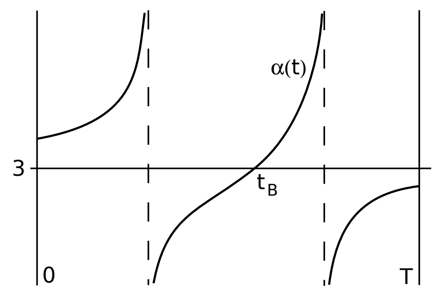

FIGURE 2. Plot of $\alpha(t)$.

Assume that $\alpha(t) \geq 3$ for $t \in (0, t_1)$ and $\alpha(t) \leq 3$ for $t \in (t_2, T)$, being the other case analogous. Then, $t_1 < t_A < t_B < t_2$ (note that $F(t,2) = 2A(t)u(t,x)$).

Denote

$$R = \{(t,\alpha) : t_1 < t < t_2, \alpha < \alpha(t)\} \cup \{(t,\alpha) : t_2 < t < T, \alpha > \alpha(t)\}.$$

Since the sign of $F(t,\alpha)$ does not change for $(t,\alpha) \in R$, and $(t_B, 2) \in R$, then

$$\operatorname{sgn} F(t,\alpha) = \operatorname{sgn} F(t_B, 2) = \operatorname{sgn} A(t_B), \qquad \text{for every } (t,\alpha) \in R.$$

Note that $\alpha(0) \geq 3 \geq \alpha(T)$.

Since $\beta(t)$ has two extrema (at t_1 and t_2), then it has a maximum and a minimum, which are greater and lower than $\bar\beta = \beta(0) = \beta(T)$. Then, there exists $t_{\bar\beta} \in (0,T)$ such that $\beta(t_{\bar\beta}) = \bar\beta$. Moreover, $t_A \in (t_1, t_2)$. Then $\operatorname{sgn} \beta'(t) = \operatorname{sgn} B(t_A)$ for $t_1 < t < t_2$. Therefore if $B(t_A) > 0$ then $\beta(t)$ has minimum at t_1 and a maximum at t_2 and if $B(t_A) < 0$ a maximum at t_1 and a minimum at t_2.

We shall distinguish three cases accordingly to the relative position of $\alpha(0)$, $\alpha(t_{\bar\beta})$, and $\alpha(T)$.

(1) $\alpha(t_{\bar\beta}) \in (\alpha(T), \alpha(0))$. Set $\alpha = \alpha(t_{\bar\beta})$, $\beta = \bar\beta$. Then $t_{\bar\beta}$ is the only common change of sign in $(0,T)$ of $F(t,\alpha)$ and $G(t,\beta)$. For $t \in (t_{\bar\beta}, T)$, $\operatorname{sgn} F(t,\alpha) = \operatorname{sgn} F(t_B, 2) = \operatorname{sgn} A(t_B)$ and $\operatorname{sgn} G(t,\beta) = \operatorname{sgn} B(t_A)$. Therefore, $\operatorname{sgn} I = \operatorname{sgn}(B(t_A)A(t_B))$.

(2) $\alpha(t_{\bar{\beta}}) > \alpha(0)$. Since $\beta(t)$ has a unique extremum at t_1, and it is strictly monotonic for $t \neq t_1$, $t \in (0, t_{\bar{\beta}})$, then there exist two continuous monotonic functions T_1, T_2, defined in (β_1, β_2) such that $\beta \circ T_k = \mathrm{Id}$, $T_1(\beta) < t_1 < T_2(\beta) < t_{\bar{\beta}}$, and T_1, T_2 have opposite monotonicity. Define the continuous function

$$d(\beta) = \alpha(T_1(\beta)) - \alpha(T_2(\beta)), \quad \beta \in (\beta_1, \beta_2).$$

Since $\alpha(t) \to \pm\infty$ as $t \to t_1^{\mp}$, then $d(\beta(t)) > 0$ for $t < t_1$ close enough. On the other hand, $d(\bar{\beta}) = \alpha(0) - \alpha(t_{\bar{\beta}}) < 0$. By continuity there exists β_0 such that $d(\beta_0) = 0$. For $\alpha = \alpha(T_1(\beta_0))$ and $\beta = \beta_0$, F, G have the same changes of sign: exactly two and both in $(0, t_{\bar{\beta}})$. Moreover, the sign of $F(t, \alpha)$ between the two zeroes is the sign of $-A(t_B)$ and the sign of G between the two zeroes is $-B(t_A)$. Therefore, the sign of I is the sign of $B(t_A)A(t_B)$.

(3) $\alpha(t_{\bar{\beta}}) < \alpha(T)$. It is analogous to previous case.

□

2.2. **Proof of Theorem 1.3.** In the following, we shall assume that (1.1) is monotonic with respect to λ for $\lambda \in [\lambda_1, \lambda_2]$, and $(C_1), (C_2)$ hold for every λ. Note there is no restriction assuming that A, B are increasing with respect to λ, since otherwise we apply the change of variables $\lambda \to -\lambda$.

Define the *displacement function* as

$$d(x, \lambda) = u(T, x, \lambda) - x,$$

for every x, λ such that $u(t, x, \lambda)$ is defined for every $t \in [0, T]$. Note that positive limit cycles are the positive zeroes of $x \to d(x, \lambda)$. Deriving with respect to λ,

$$d_\lambda(x, \lambda) = u_\lambda(T, x, \lambda) > 0,$$

where we have used the monotonicity with respect to λ. Therefore, for each $x > 0$ there is at most one $\lambda(x) \in [\lambda_1, \lambda_2]$ such that

$$d(x, \lambda(x)) = 0.$$

Let D be the domain of definition of the function $\lambda(x)$. Since $d_\lambda(x, \lambda) > 0$, if $\lambda_1 < \lambda < \lambda_2$, then $\lambda(x)$ coincides with the unique solution of $d(x, \lambda) = 0$ defined in a neighborhood of x by the Implicit Function Theorem.

Thus, because $N(\lambda_1)$ is finite, either D is empty, or $D = \mathbb{R}^+$, or there exist $n \in \mathbb{N}$ and disjoints intervals I_j such that $D = I_0 \cup I_2 \cup \cdots \cup I_{2n}$, where $I_0 = (0, x_1]$, $x_1 > 0$, $I_{2j} = [x_{2j}, x_{2j+1}]$, $j = 1, \ldots, n-1$, and

$$I_{2n} = \begin{cases} [x_{2n}, x_{2n+1}], & \text{if} \quad x_{2n+1} < \infty, \\ [x_{2n}, \infty), & \text{otherwise,} \end{cases}$$

or $D = I_1 \cup I_3 \cup \cdots \cup I_{2n+1}$, where $I_{2j-1} = [x_{2j-1}, x_{2j}]$, $j = 1, \ldots, n$, and

$$I_{2n+1} = \begin{cases} [x_{2n+1}, x_{2n+2}], & \text{if} \quad x_{2n+2} < \infty, \\ [x_{2n+1}, \infty), & \text{otherwise.} \end{cases}$$

Then, for every λ, the initial conditions of positive limit cycles are determined by the solutions of $\lambda(x)=\lambda$.

Moreover, deriving with respect to x in $d(x,\lambda(x))=0$,

$$d_x(x,\lambda(x))+d_\lambda(x,\lambda(x))\lambda'(x)=0.$$

And, if x is a zero of $\lambda'(x)$,

$$d_{xx}(x,\lambda(x))+2d_\lambda(x,\lambda(x))\lambda''(x)=0$$

Therefore, $u_{xx}(T,x,\lambda)$ and $\lambda''(x)$ have opposite sign.

Now, as $(C_1),(C_2)$ hold for every $\lambda\in(\lambda_1,\lambda_2)$, then $(t_B-t_A)B(t_A)A(t_B)\neq 0$ for every $\lambda\in(\lambda_1,\lambda_2)$, and by continuity the sign of $(t_B-t_A)B(t_A)A(t_B)$ is constant. We shall assume $(t_B-t_A)B(t_A)A(t_B)$ has negative sign, as the positive case follows analogously. Then, for every interior point $x\in D$ such that $\lambda'(x)=0$, we have $\lambda''(x)>0$. Hence, $\lambda(x)$ has only minimum interior points. Therefore, $\lambda(x)$ has at most one minimum interior point in each I_j. Since two consecutive hyperbolic limit cycles have opposite stability, $\lambda(x)$ is an "alternate" monotonic function in D minus the set of minimum interior points. Moreover $\lambda(x_j)$ is equal to λ_1 or λ_2 for each finite x_j.

In consequence, in every interval I_j, $\lambda(x)=\lambda$ has at most two solutions for $\lambda\in[\lambda_1,\lambda_2]$. Thus, we have a maximum of $2n+2=N(\lambda_2)+2$ solutions. If no closed solutions bifurcate from the origin, then $D=I_1\cup I_3\cup\cdots\cup I_{2n-1}$, so $\lambda(x)=\lambda_2$ has at most $2n$ or $2n+1$ solutions depending on whether I_{2n+1} is a bounded or unbounded interval.

3. Example of application

Consider the Abel equation (1.2), where $A(t)=at(t-t_A)$ and $B(t)=b(t-t_B)$. As we have explained in the introduction we can restrict our study to the case $0<t_B<t_A<1$. Next result gives a sufficient algebraic condition to verify when hypothesis (C_2) of Theorem 1.2 holds, so that it can be checked computationally.

Proposition 3.1. *Suppose that (C_1) holds and $0<t_B<t_A<1$. Let $J=(\tau_1,\tau_2)$ be each one of the intervals in which t_B,t_A divide $(0,1)$.*

Assume that the function

$$P(t,\gamma)=4(B(t)A'(t)-B'(t)A(t))+B^3(t)\gamma(\gamma-2) \tag{3.7}$$

satisfies one of the following conditions:

1. *$P(t,\gamma)$ has no zeroes in J for every γ with the same sign as ϕ.*
2. *$P(t,\gamma)$ has exactly one zero in J for every γ with the same sign as ϕ, and either $\phi(\tau_1)=0$, $\phi(\tau_2)=\pm\infty$, or $\phi(\tau_1)=\pm\infty$, $\phi(\tau_2)=0$.*

Then $u(t,x)-\gamma\phi(t)$ has at most one simple zero in J for any γ.

Proof. Firstly,

$$\begin{aligned}
&(\gamma\phi)' - B(\gamma\phi)^2 - A(\gamma\phi)^3 \\
&\quad = \gamma(\phi' - B\gamma\phi^2 - A\gamma^2\phi^3) \\
&\quad = \gamma\left(\frac{BA' - B'A}{2A^2} - \frac{\gamma B^3}{4A^2} + \frac{\gamma^2 B^3}{8A^2}\right) \\
&\quad = \frac{\gamma}{8A^2}(4(BA' - B'A) + B^3\gamma(\gamma - 2)).
\end{aligned}$$

If γ and ϕ have opposite signs then $\gamma\phi$ is negative and therefore it does not intersects the positive solution u.

If $(\gamma\phi)'(t) - B(t)(\gamma\phi)^2 - A(t)(\gamma\phi)^3$ has no zeroes in (τ_1, τ_2), then $\gamma\phi$ is an upper or lower solution of u and therefore, it intersects in at most one point.

If $(\gamma\phi)'(t) - B(t)(\gamma\phi)^2 - A(t)(\gamma\phi)^3$ has at most one zero in (τ_1, τ_2), then $(\gamma\phi)$ changes from upper (resp. lower) solution to lower (resp. upper) solution of (1.1) in that interval. Since $u(t, x)$ is bounded, it intersects to $\gamma\phi$ at one point in (τ_1, τ_2) (note that, if $0 < \tau_1, \tau_2 < T$, $\gamma\phi$ divides the region $u > 0$ into two, and $u(t, x)$ must change of region an even number of times). □

Proposition 3.2. *For any $a, b > 0$, $0 < t_B < t_A < 1$, condition (C_2) holds.*

Proof. First, we need to compute the zeroes of $P(t, \gamma)$ and determine its relative position with respect to t_B, t_A, and $\gamma = 0$. The discriminant of $P(t, \gamma)$ with respect to t is:

$$\mathrm{Dis}(P(t,\gamma), t) = 16a^2b^4(t_B - t_A)t_B(64a^2 + 27b^4(t_A - t_B)t_B(\gamma - 2)^2\gamma^2) < 0.$$

Therefore, for every fixed γ, if the number of zeroes of $P(t, \gamma)$ in $(0, 1)$ changes it must be by through a zero of $P(0, \gamma)$ or $P(1, \gamma)$. We have that

$$P(0, \gamma) = bt_B\left(4at_A + b^2t_B^2\gamma(2 - \gamma)\right).$$

Then $P(0, \gamma) = 0$ is the graph of the function

$$h_0(b, t_B, t_A, \gamma) = \frac{b^2t_B^2\gamma(\gamma - 2)}{4t_A} \quad \text{defined for } b > 0,\ 0 < t_B < t_A < 1.$$

which is positive when $\gamma < 0$ or $\gamma > 2$. Therefore, it divides the space of parameters in three connected components, R_1, R_2, R_3 such that $\gamma < 0$ for the points in R_1, and $\gamma > 2$ for the points in R_3, see Figure 3.

Since

$$P(1, \gamma) = b\left(4a(1 + (t_A - 2)t_B) + b^2(t_B - 1)^3(2 - \gamma)\gamma\right),$$

then $P(1, \gamma) = 0$ is the graph of the function

$$h_1(b, t_B, t_A, \gamma) = \frac{b^2(t_B - 1)^3(\gamma - 2)\gamma}{4(1 + (t_A - 2)t_B)} \quad \text{defined for } b > 0,\ 0 < t_B < t_A < 1.$$

Since $t_A > t_B$, $1 + (t_A - 2)t_B > 1 + (t_B - 2)t_B > 0$. Then h_1 is positive when $0 < \gamma < 2$. Then, it divides R_2 into two connected regions, R_2^-, R_2^+, such

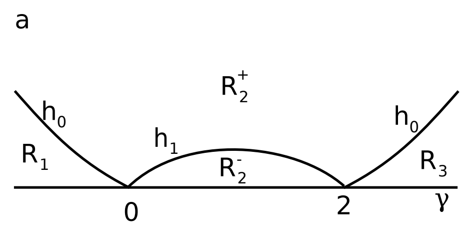

FIGURE 3. Regions for fixed b, t_B, t_A

that R_2^- are the points of R_2 such that $0 < a < h_1(b, t_B, t_A, \gamma)$, and R_2^+ are the points of R_2 such that $a > h_1(b, t_B, t_A, \gamma)$.

To compute the relative position of the zeroes of P and t_B, we simply compute the resultant of P and B (considered as polynomials in t), obtaining

$$\operatorname{Res}(P, B, t) = 4ab^4 t_B(t_B - t_A) < 0.$$

So the relative position does not change in any region.

Finally, it suffices to fix one value of each of the parameters in each region and compute the zeroes of P for these parameters. We fix $t_A = 3/4$, $t_B = 2/5$, $b = 1$, and choose for the region R_1, $a = 1/10$, $\gamma = -2$, for the region R_2^+, $a = \gamma = 1$, for the region R_2^-, $a = 1/10$, $\gamma = 1$, and for the region R_3, $a = 1/10$, $\gamma = 33/10$. For each of these values we compute the number on zeroes of $P(t, \gamma)$ for $t \in (0, 1)$, obtaining:

(1) In R_1, R_3, $P(t, \gamma)$ has a unique zero, $\bar{t}$, and $0 < \bar{t} < t_B$.
(2) In R_2^-, $P(t, \gamma)$ has a unique zero, $\bar{t}$, and $t_B < \bar{t} < 1$.
(3) In R_2^+, $P(t, \gamma)$ has no zeroes.

To conclude, we apply Proposition 3.1 to the intervals $(0, t_B)$, (t_B, t_A), and $(t_A, 1)$.

For $\gamma > 0$, $\gamma\phi$ is positive only in the interval (t_B, t_A). We have $\phi(t_B) = 0$, $\phi(t_A) = \infty$, and $P(t, \gamma)$ has at most one simple zero such that $\bar{t} > t_B$, so Proposition 3.1 holds, and $u - \gamma\phi = 0$ at at most one point.

For $\gamma < 0$, $\gamma\phi$ is positive in the intervals $(0, t_B)$, and $(t_A, 1)$. We have $\phi(0) = +\infty$, $\phi(t_B) = 0$, and $P(t, \gamma)$ has at most one zero $\bar{t}$ such that $\bar{t} < t_B$. So Proposition 3.1 holds for every interval, and $u - \gamma\phi = 0$ has at most two simple zeroes in $(0, 1)$. □

Proof of Theorem 1.1. We study the stability of the origin following the procedure used in [5] and based on the approach of [2, 3] adapted to our special case. The first Lyapunov constant is

$$\int_0^1 B(t)\, dt = 2\Big(\frac{1}{2} - t_B\Big) b.$$

When it is is equal to zero, then the next Lyapunov constant is

$$\int_0^1 A(t)\,dt = \frac{3}{2}\Big(\frac{2}{3} - t_A\Big)a$$

and if this is also equal to zero, then the stability is given by the sign of

$$\int_0^1 A(t)\int_0^t B(s)\,ds\,dt = \frac{23}{180}ab.$$

If $t_A = 2/3$ and $t_B = 1/2$, then $x = 0$ is a solution of (1.2) with multiplicity 4, and the origin is unstable. Taking $t_A > 2/3$, close to $2/3$, we change the stability of the origin and generate a simple limit cycle by a Hopf-like bifurcation. Taking $t_B < 1/2$, close to $1/2$, we generate a second limit cycle. So the upper bound of two positive limit cycles is sharp. Let us prove that it is an upper bound.

Denote by $N(t_B, t_A)$ the maximum number of positive limit cycles of the corresponding Abel equation (1.2). As we have shown in the introduction, unless $0 < t_B < t_A < 1$, we already know that $N(t_B, t_A) \leq 1$. In particular, $\max\big(N(t_B, 0), N(t_B, 1)\big) \leq 1$ and moreover $\max\big(N(1/2, 0), N(1/2, 1)\big) = 0$. Fix $t_B \neq 1/2$. Hence no limit cycles bifurcate from $x = 0$ and by Theorem 1.3, using t_A as a monotonicity parameter, we get that $N(t_B, t_A) \leq \max\big(N(t_B, 0), N(t_B, 1)\big) + 1 \leq 2$. When $t_B = 1/2$, applying again the same theorem, we get $N(1/2, t_A) \leq \max\big(N(1/2, 0), N(1/2, 1)\big) + 2 = 2$, as we wanted to prove, see Figure 1. □

3.1. **Stability at infinity.** To complete the bifurcation diagram, we need to obtain the possible bifurcations at infinity. To simplify the notation, we shall assume $a = b = 1$, but the results hold for $a, b > 0$.

By the change of variables $y = x^{-1}$, (1.1) becomes

$$y' = dy/dt = -B(t) - A(t)y^{-1}. \tag{3.8}$$

Note that for $y > 0$, the portrait of the integral curves of (3.8) and the phase plane of

$$\dot t = dt/ds = y, \quad \dot y = dy/ds = -B(t)y - A(t), \tag{3.9}$$

where s is a new time variable, are the same. This approach is also used in [15, 16].

The Jacobian matrix of (3.9) at the equilibrium point $(0,0)$ is

$$\begin{pmatrix} 0 & 1 \\ -A'(0) & -B(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t_A & t_B \end{pmatrix}.$$

The eigenvalues of the jacobian matrix are $\lambda^\pm = \Big(t_B \pm \sqrt{t_B^2 + 4t_A}\Big)/2$. Since $(0,0)$ is a saddle point, there exists a unique analytic invariant unstable manifold, tangent to $< (1, \lambda^+) >$ at $(0,0)$. The branch of the manifold in $\{(t,y)\colon t > 0,\ y > 0\}$ is defined by a solution of (3.9) such that

$$\lim_{s\to-\infty}(t(s), y(s)) = (0,0), \qquad \lim_{s\to-\infty}\frac{y'(s)}{t'(s)} = \lambda^+.$$

Thus, there exists a unique analytic solution of (3.8), $v^\infty(t)$ defined in an interval $(0,\beta)$ such that

$$\lim_{t\to 0^+} v^\infty(t)=0,\quad \lim_{t\to 0^+} v^\infty_t(t)=\lambda^+.$$

Since $(1,0)$ it is a regular point, there exists a unique solution of (3.9) determined by this initial condition. The intersection of the solution with $y>0$ determines a positive solution $w^\infty(t)$ of (3.8) such that

$$\lim_{t\to 1^-} w^\infty(t)=0,\quad \lim_{t\to 1^-} w^\infty_t(t)=\infty.$$

By continuation of solutions, v^∞ or w^∞ is defined for every $t\in(0,1)$. Moreover, by uniqueness of solutions, both are defined in $(0,1)$ if and only if they coincide.

Note that a positive solution (1.2) corresponds after the change of variables to a positive solution of (3.8). Moreover, the set of bounded solutions of (1.2) is limited by either $v^\infty(t)$ or $w^\infty(t)$. If we have a continuous family of closed solutions of (1.2) $u(t,\lambda)$, depending on a parameter λ, and

$$\lim_{\lambda\to\lambda^*}\sup_{t\in(0,1)} u(t,\lambda)=\infty,$$

then for the parameter λ^*, v^∞ and w^∞ coincide.

We say (t_B,t_A) is a bifurcation value at infinity if v^∞ and w^∞ coincide. If (t_B,t_A) is a bifurcation value at infinity, the Poincaré map of (3.8) is defined for $\tau>0$ as

$$P(\tau)=v(1,\tau),$$

where $v(t,\tau)$ is the solution of (3.8) determined by the initial condition $v(0,\tau)=\tau$, see also Figure 4.

Proposition 3.3. *Assume that (t_B,t_A) is a bifurcation value at infinity. Then, for these values of the parameters, the sign of $P(\tau)-\tau$ is the sign of t_B for every $\tau>0$ small enough. Consequently, the stability of infinity is given by the sign of t_B.*

Proof. To determine the sign of $P(y_0)-y_0$ for $y_0>0$ close to 0, we consider four sections (see Figure 4): $S_0=\{(0,y_0)\colon y\ge 0\}$, S_1 which will be defined below, $S_2=\{(t,\delta)\colon v^\infty(t)\le\delta, t<1\}$ depending on $\delta>0$, $S_3=\{(1,y)\colon y\ge 0\}$, and define the Poincaré maps P_i from S_{i-1} to S_i, $1\le i\le 3$. Therefore, $P=P_3\circ P_2\circ P_1$.

Consider (3.8) in a neighborhood of the singular point $(0,0)$. By a linear change of variables, we may write the system as

$$\dot x=\lambda^+x+O(x^2,xz,z^2),\quad \dot z=\lambda^-z+O(x^2,xz,z^2), \tag{3.10}$$

and S_0 becomes $\{(x,rx)\colon x>0\}$, for some $r>0$.

Now by the change of variables $\Phi(x,z)=(x-\psi(z),z-\phi(x))$ (see [20, p. 49]), where $(x,\phi(x)),(\psi(z),z)$ are the graphs of the unstable and stable varieties, respectively, (3.10) becomes

$$\dot x=\lambda^+x+xO(x,z),\quad \dot z=\lambda^-z+zO(x,z). \tag{3.11}$$

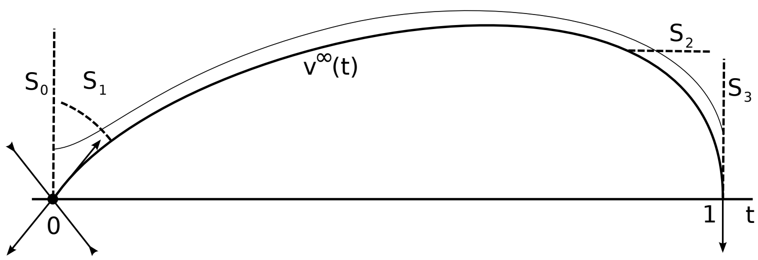

FIGURE 4. Sections for the Poincaré map close to $v^\infty(t)$

Now, $z = 0, x = 0$ are the unstable and stable varieties, respectively, and $\tilde{S}_0 = \{(x, rx + O(x^2))\colon x > 0\}$ is the image by Φ of $\{(x, rx)\colon x > 0\}$.

For any $\delta > 0$, we define $\tilde{S}_1 = \{(\delta, z)\colon z \geq 0\}$, and S_1 to be the transversal section of (3.9) preimage of $\tilde{S}_1$ by the change of variables above.

For every $\epsilon > 0$, there exists $\delta > 0$, such that if $\tilde{P}_{(1,\epsilon)}(\tau)$ is the Poincaré map from $\tilde{S}_0$ to $\tilde{S}_1$ for the linear system

$$\dot{x} = (\lambda^+ + \epsilon)x, \quad \dot{z} = \lambda^- z \tag{3.12}$$

then $\tilde{P}_{(1,\epsilon)}(\tau) < \tilde{P}_1(\tau)$, where $\tilde{P}_1(\tau)$ denotes the Poincaré map from $\tilde{S}_0$ to $\tilde{S}_1$ for (3.11), see again [20].

Let $(x(s), z(s))$ be the solution of (3.12) determined by the initial condition $(\tau, r\tau + O(\tau^2)) \in \tilde{S}_0$. Then

$$x(s) = \tau \exp(s(\lambda^+ + \epsilon)), \quad z(s) = \big(r\tau + O(\tau^2)\big)\exp(s\lambda^-).$$

Denote $s_1 > 0$ the first time such that $(x(s_1), z(s_1)) \in \tilde{S}_1$. Then,

$$\tau \exp(s(\lambda^+ + \epsilon)) = \delta, \quad (r\tau + O(\tau^2))\exp(s\lambda^-) = \tilde{P}_{(1,\epsilon)}(\tau).$$

Since

$$\exp(s(\lambda^+ + \epsilon)\lambda^-) = \delta^{\lambda^-}\tau^{-\lambda^-},$$

then

$$(r\tau + O(\tau^2))^{\lambda^+ + \epsilon}\delta^{-\lambda^-}\tau^{-\lambda^-} = \tilde{P}^{\lambda^+ + \epsilon}_{(1,\epsilon)}(\tau)$$

and

$$\tilde{P}_{(1,\epsilon)}(\tau) = \left(r\tau^{\frac{\lambda^+ + \epsilon - \lambda^-}{\lambda^+ + \epsilon}} + O\left(\tau^{1 + \frac{\lambda^+ + \epsilon - \lambda^-}{\lambda^+ + \epsilon}}\right)\right)\delta^{-\frac{\lambda^-}{\lambda^+ + \epsilon}}.$$

Arguing analogously with

$$\dot{x} = (\lambda^+ - \epsilon)x, \quad \dot{z} = \lambda^- z, \tag{3.13}$$

we obtain

$$\tilde{P}_{(1,\epsilon)}(\tau) < \tilde{P}_1(\tau) < \tilde{P}_{(1,-\epsilon)}(\tau) = \left(r\tau^{\frac{\lambda^+ - \epsilon - \lambda^-}{\lambda^+ - \epsilon}} + O\left(\tau^{1 + \frac{\lambda^+ - \epsilon - \lambda^-}{\lambda^+ - \epsilon}}\right)\right)\delta^{-\frac{\lambda^-}{\lambda^+ - \epsilon}}.$$

Since the change of variables $\Phi(x,z)$ preserves the linear part at $(0,0)$ of the differential system, if we denote $P_{(1,\epsilon)}, P_1, P_{(1,-\epsilon)}$ the corresponding Poincaré maps in (3.8), the lowest exponent of τ is the same.

Since from S_1 to S_2 the graph of v^∞ has only regular points, $P_2(\tau) = K\tau + O(\tau^2)$, for certain constant $K > 0$.

Differential equation (3.8) in a neighborhood the point $(1,0)$ can be written as

$$\frac{dt}{dy} = \frac{y}{t_A - 1} + O(y^2). \tag{3.14}$$

Integrating in $(0,y)$,

$$t(y) = C + \frac{y^2}{2(t_A - 1)} + O(y^3)$$

The map from S_2 to $\{(t,0)\colon t \geq 1\}$ is of the from $\tau \to k\tau + O(\tau^2)$, for a positive constant k. Then $C = 1 + k\tau + O(\tau^2)$. Now, from $t(P_3(\tau)) = 1$,

$$P_3(\tau) = \sqrt{2k(1 - t_A)\tau} + O(\tau).$$

Composing the three maps, we obtain

$$K_1\tau^{\frac{\lambda^+ + \epsilon - \lambda^-}{2(\lambda^+ + \epsilon)}} + O\left(\tau^{\frac{2(\lambda^+ + \epsilon) - \lambda^-}{2(\lambda^+ + \epsilon)}}\right) < P(\tau) < K_2\tau^{\frac{\lambda^+ - \epsilon - \lambda^-}{2(\lambda^+ - \epsilon)}} + O\left(\tau^{\frac{2(\lambda^+ - \epsilon) - \lambda^-}{2(\lambda^+ - \epsilon)}}\right),$$

for certain constants $K_1, K_2 > 0$. Taking $\epsilon > 0$ sufficiently small, we get that

$$\operatorname{sgn}\big(P(\tau) - \tau\big) = \operatorname{sgn}\left(1 - \frac{\lambda^+ - \lambda^-}{2\lambda^+}\right) = \operatorname{sgn}(\lambda^+ + \lambda^-) = \operatorname{sgn}(t_B).$$

$\square$

Remark 3.4. Following the same steps that in the proof of Proposition 3.3 we get that if a smooth planar vector field $\dot{\mathbf{x}} = X(\mathbf{x})$, with $X(\mathbf{0}) = \mathbf{0}$, presents in its phase portrait a situation as the one given in Figure 4, then $\operatorname{sgn}(P(\tau) - \tau) = \operatorname{sgn}(\operatorname{div}(X(\mathbf{0})))$, where as usual div denotes the divergence. We only need to assume that the origin is a hyperbolic saddle and that the contact point of the unstable separatrix with the (non-transversal) section S_3 is quadratic. Notice that the divergence at the origin is precisely $\mu^+ + \mu^-$, where $\mu^- < 0 < \mu^+$ are its associated eigenvalues. The intuition behind the proof is that the Poincaré map P is the composition of three maps with respective dominant terms, $C_1\tau^{\frac{\mu^+ - \mu^-}{\mu^+}}$, $C_2\tau$ and $C_3\sqrt{\tau}$, for some positive real constants, $C_i, i = 1,2,3$.

Observe also that if the contact point of the separatrix with S_3 is of k-th order, $k > 2$, then

$$\operatorname{sgn}\big(P(\tau) - \tau\big) = \operatorname{sgn}\left(1 - \frac{\mu^+ - \mu^-}{k\mu^+}\right) = \operatorname{sgn}((k-1)\mu^+ + \mu^-),$$

which is no more the divergence of X at the saddle point. The reason is that in this situation the third map has dominant term $C_3\tau^{1/k}$.

3.2. **Bifurcation diagram.** For equation (1.2), Theorem 1.2 determines the saddle-node bifurcations, Proposition 3.3 determines the bifurcations at infinity, and the bifurcations at the origin were studied in the proof of Theorem 1.1. Therefore, we can obtain its bifurcation diagram. To simplify the analysis, we shall assume $a = b = 1$.

The following results describe the infinity and saddle-node bifurcation curves.

Proposition 3.5. *There exists a strictly-decreasing analytic function T_∞ defined in $[0,1]$ such that for every $t_B \in [0,1]$, $(t_B, T_\infty(t_B))$ is a bifurcation value at infinity.*

Proof. Fix $t_B \in [0,1]$ and assume (t_B, t_A) is a bifurcation value at infinity of (1.2). Since (1.2) is monotonic with respect to t_A, this bifurcation value has to be unique. Therefore, we define $T_\infty(t_B) = t_A$.

Since (1.2) is monotonic with respect to t_B, then $T_\infty(t_B)$ is strictly decreasing. Let $(t^\infty(s, t_A, t_B), y^\infty(s, t_A, t_B))$ denote the positive branch of the unstable variety of (3.9). Since $T_\infty(t_B)$ is determined by $t^\infty(s, t_A, t_B) = 1$, $y^\infty(s, t_A, t_B) = 0$, $T_\infty(t_B)$ is an analytic function.

To show that T_∞ is defined for all $t_B \in [0,1]$, we only need to prove that $0 < T_\infty(0), T_\infty(1) < 1$. This is the most difficult part. We defer this proof to Proposition 3.9.

□

Remark 3.6. If a, b are different from one, then it is no longer always true that the function T_∞ is well defined on the whole interval $[0,1]$.

Proposition 3.7. *For every $t_B \in (0, 1/2)$, there exists a unique value $t_A \in (0,1)$ such that (1.1) has a saddle-node bifurcation (semistable limit cycle). Moreover, these bifurcations values are given by a strictly-decreasing analytic function T_d, such that $T_d(t_B) < T_\infty(t_B)$.*

Proof. The domain of definition of $T_d(t_B)$ is included in the interval $(0, 1/2)$. Indeed, for $t_B > 1/2$,

$$u(1, x, t_B, t_A) - x < 0 \text{ for } x > 0 \text{ close to zero.}$$

Thus, if there exist two positive limit cycles for (t_B, t_A), then there exists a singular limit cycle for $(t_B, \bar{t}_A)$, where $1 > \bar{t}_A > t_A$, with $u_{xx}(1, t_B, \bar{t}_A) < 0$, which is not possible by Theorem 2.1. The case $t_B = 1/2$ follows similarly.

Let $0 < t_B < 1/2$. Then

$$u(1, x, t_B, t_A) - x > 0 \text{ for } x > 0 \text{ close to zero.}$$

By Theorem A of [11], (1.2) has no positive limit cycles for $t_A = 0$ and $0 \le t_B \le 1/2$.

By Proposition 3.3,

$$u(1, x, t_B, T_\infty(t_B)) - x < 0 \text{ for } x > 0 \text{ close to } \infty.$$

Therefore, there exists a simple positive limit cycle. Since for every $0 < t_A < T_\infty(t_B)$ there is no bifurcation at the origin or at infinity, there exist $x > 0$, $0 < \bar{t}_A < T_\infty(t_B)$ such that

$$u(t, x, t_B, \bar{t}_A) = x, \quad u_x(t, x, t_B, \bar{t}_A) = 0.$$

We define $T_d(t_B) = \bar{t}_A$. Applying the Implicit Function Theorem to

$$u(1, x, t_B, t_A) - x = 0, \quad u_x(1, x, t_B, t_A) - 1 = 0,$$

using that singular limit cycles satisfy

$$u_{xx}(1, x, t_B, T_d(t_B)) > 0,$$

and that (1.2) is monotonic with respect to t_A, t_B, we obtain the existence of analytic functions $x(t_B), T_d(t_B)$ such that

$$u(1, x(t_B), t_B, T_d(t_A)) - x(t_B) = 0, \quad u_x(1, x(t_B), t_B, T_d(t_B)) - 1 = 0.$$

Moreover, $T_d(t_B)$ is strictly decreasing. $\square$

Using results above, it is not difficult to prove the following result.

Theorem 3.8. *Consider Abel equation* (1.2) *with* $a = b = 1$. *Define the regions:*

$$\begin{aligned}
R_1 =&\{(t_B, t_A) \colon 1/2 < t_B < 1,\ T_\infty(t_B) < t_A \leq 1\},\\
R_2 =&\{(t_B, t_A) \colon 0 < t_B < 1/2,\ 0 \leq t_A < T_d(t_B)\},\\
R_3 =&\{(t_B, t_A) \colon 1/2 < t_B < 1,\ 0 \leq t_A < T_\infty(t_B)\},\\
R_4 =&\{(t_B, t_A) \colon 0 < t_B < 1/2,\ T_\infty(t_B) < t_A \leq 1\},\\
R_5 =&\{(t_B, t_A) \colon 0 < t_B < 1/2,\ T_d(t_B) < t_A < T_\infty(t_B)\}.
\end{aligned}$$

Then

(1) *For every* $(t_B, t_A) \in R_1 \cup R_2$, (1.2) *has no positive limit cycles.*
(2) *For every* $(t_B, t_A) \in R_3 \cup R_4$, (1.2) *has exactly one positive simple limit cycle.*
(3) *For every* $(t_B, t_A) \in R_5$, (1.2) *has two positive simple limit cycles.*

Previous description can be summarized in the bifurcation diagram given in Figure 1. Moreover,

(1) *On the curve* T_d, (1.2) *has exactly one positive double limit cycle.*
(2) *On the curve* T_∞, *if* $0 \leq t_B < 1/2$, (1.2) *has exactly a positive simple limit cycle at* $(t_B, T_\infty(t_B))$, *and no positive limit cycles otherwise.*
(3) *On the straight line* $t_B = 1/2$, (1.2) *has exactly one positive simple limit cycle when* $t_A \in (2/3, T_\infty(1/2))$ *and no positive limit cycles otherwise.*

Next result concludes the proof of Proposition 3.5.

Proposition 3.9. *Consider Abel equation* (1.2) *with* $a = b = 1$ *and the function* T_∞ *introduced in Propositions 3.5. Then,* $0.13 < T_\infty(1) < 1$ *and* $0 < T_\infty(0) < 0.93$ *and therefore* $[0, 1]$ *is in the interval of definition of* T_∞.

Proof. Set $t_B = 1$. Let $v_3(t, t_B, t_A)$ be the Taylor polynomial of degree three of $v^\infty(t, 1, t_A)$. Note that v^∞ may be computed by writing

$$v^\infty(t, t_B, t_A) = \sum_{k=1}^{\infty} c_k t^k,$$

and computing the first coefficients c_k imposing that v^∞ satisfies (3.8). This can be done computed aided and we obtain that

$$v_3(t) = \frac{1}{2} K\, t - \frac{(t_A + K)}{3t_A + K}\, t^2 - \frac{2\,(t_A + K)\left(-2t_A^2 + 2K + t_A\,(4 + K)\right)}{K\,(3t_A + K)^2\,(8t_A + 3K)}\, t^3,$$

where $K = 1 + \sqrt{1 + 4t_A}$.

Next, let us choose $\tilde{t}_A$ such that $v_3(1) = 0$. We can algebraically determine that such $\tilde{t}_A > 0.13$. Finally, we study the sign of

$$s(t) = v_3'(t) + A(t)/v_3(t) + B(t).$$

Since $s(t) > 0$, then $v_3(t)$ is an upper solution of (3.8) and therefore, $v_3(t) > v^\infty(t)$ for $t_A = \tilde{t}_A$. Finally, since the right hand side of (3.8) is strictly increasing with respect to t_A, $v_3(t) > v^\infty(t)$ for any $0 < t \le 1$ (whenever defined), $t_A < \tilde{t}_A$, and, therefore, $\tilde{t}_A < T_\infty(1)$.

To obtain the second bound, set $t_B = 0$, and consider again system (3.9). Let $(t(s), y(s))$ be the solution determined by the initial condition $t(0) = 1$, $y(0) = 0$. Now, we compute the Taylor polynomial of t and y of degree four. Denote them by t_4 and y_4, respectively. Now we obtain $\hat{t}_A$ such that there exists $\hat{s} < 0$ such that $t_4(\hat{s}) = y_4(\hat{s}) = 0$. The value $\hat{t}_A$ is computed as a zero of the resultant of t_4 and y_4 with respect to s. It can be algebraically checked that $0 < \hat{t}_A < 0.93$. Next we compute $\hat{s}$ solving the system $t_4(\hat{s}) = y_4(\hat{s}) = 0$.

As we check that the field change of orientation with respect to the curve $(t_4(s), y_4(s))$, we modify y_4 by subtracting $(s - \hat{s})/100$. Then it can be shown that the field has derivative always pointing to the upper side of $(t_4(s), y_4(s))$. Then the solution starting in $(0, 0)$ satisfies $y(1) > 0$, and therefore $T_\infty(0) < \hat{t}_A < 0.93$.

□

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