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STABILITY OF SINGULAR LIMIT CYCLES FOR ABEL EQUATIONS

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Abstract. We obtain a criterion for determining the stability of singular limit cycles of Abel equations $x' = A(t)x^3 + B(t)x^2$. This stability controls the possible saddle-node bifurcations of limit cycles. Therefore, studying the Hopf-like bifurcations at $x = 0$, together with the bifurcations at infinity of a suitable compactification of the equations, we obtain upper bounds of their number of limit cycles. As an illustration of this approach, we prove that the family $x' = at(t - t_A)x^3 + b(t - t_B)x^2$, with $a, b > 0$, has at most two positive limit cycles for any t_B, t_A.

1. **Introduction and main results.** The study of the number of periodic solutions of Abel differential equations is a challenging question. These equations are interesting because they provide models of real phenomena, see for instance [4, 9, 12], or as a tool for studying several subcases of Hilbert XVI problem on the number of limit cycles of planar polynomial differential equations, see [7, 15].

In this paper we consider Abel equations,

$$\frac{dx}{dt} = x' = A(t)x^3 + B(t)x^2, \tag{1}$$

with $A(t), B(t)$ continuous functions defined on $[0, T]$. Let $u(t, x)$ denote the solution of (1) determined by $u(0, x) = x$. We say $u(t, x)$ is *closed* or *periodic*, if $u(T, x) = x$, and *singular* or *multiple*, if it is closed and $u_x(T, x) = 1$. When $u_x(T, x) \neq 1$ then it is said that it is *simple* or *hyperbolic*. Isolated closed solutions are also called *limit cycles* and a singular closed solution such that $u_{xx}(T, x) \neq 0$ will be called a *double closed solution*, or also a *semistable limit cycle*.

Notice that $x = 0$ is always a closed solution of (1). Therefore the number of limit cycles in regions $x > 0$ and $x < 0$ can be studied separately. Since one region can be sent to the other one with the transformation $x \to -x$, we will restrict our attention to the region $x > 0$.

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