A NEW QUALITATIVE PROOF OF A RESULT ON THE REAL JACOBIAN CONJECTURE

FRANCISCO BRAUN[1] AND JAUME LLIBRE[2]

Abstract. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF(x)$ is different from zero for all $x \in \mathbb{R}^2$. We assume that the degrees of f and g are equal. We denote by $\overline{f}$ and $\overline{g}$ the homogeneous part of higher degree of f and g, respectively. In this note we provide a proof relied on qualitative theory of differential equations of the following result: If $\overline{f}$ and $\overline{g}$ do not have real linear factors in common, then F is injective.

1. Introduction and statement of the main result

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that $\det DF$ is nowhere zero. It is clear that F is a local diffeomorphism, but it is not always injective. There are very general well known conditions to ensure that F is a global diffeomorphism, for instance F is a global diffeomorphism if and only if it is proper (i.e. if inverse images of compact subsets are compact), or F is a diffeomorphism if and only if $\displaystyle\int_0^\infty \inf_{|x|\le s} \left\| DF(x)^{-1} \right\|^{-1} ds = \infty$. These conditions are due to Banach–Mazur and Hadamard, respectively, and remain true in more general spaces, for details, see (Plastock 1974). Another condition, now specifically of $\mathbb{R}^2$ and ensuring just the injectivity of F, is the following sufficient condition: the real eigenvalues of $DF(x)$, for all $x \in \mathbb{R}^2$, are not contained in an interval of the form $(0, \varepsilon)$, for some $\varepsilon > 0$, see (Fernandes et al. 2007) and (Cobo et al. 2002).

Now, if F is a polynomial map, the statement that F is injective is known as the *real Jacobian conjecture*. This conjecture is false, since Pinchuk constructed, in (Pinchuk 1994), a non injective polynomial map with nonvanishing Jacobian determinant. So it is natural to ask for additional conditions in order for this conjecture to hold. In (Braun and dos Santos Filho 2010), for example, it is shown that it is enough

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to assume that the degree of f is less or equal to 3. On the other hand, if we assume that $\det DF(x) = \text{constant} \neq 0$, then to know if F is injective is an open problem largely known as the *Jacobian conjecture*, see (Bass et al. 1982) and (Van den Essen 2000) for details and for surveys on the Jacobian conjecture).

In the next theorem, we provide another sufficient condition for the validity of the real Jacobian conjecture. Our result is not new. Indeed it is a consequence of the main result of (Cimma et al. 1996). But our proof is a very elementary dynamical one and relies on qualitative theory of differential equations.

Before the statement of the theorem, we introduce some notations. Given a polynomial map $p : \mathbb{R}^2 \to \mathbb{R}$, we write $\overline{p}$ the homogeneous part of higher degree of p. If $q : \mathbb{R}^2 \to \mathbb{R}$ is another polynomial map, we say that p and q do not have linear factors in common when there is not a linear polynomial $ax + by$ which divides both p and q.

Theorem 1. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero. Assume that the degrees of f and g are equal and greater than one. Then the following condition is sufficient for the global injectivity of F: the homogeneous polynomials $\overline{f}$ and $\overline{g}$ do not have real linear factors in common.*

The following map shows that the condition in Theorem 1 is not necessary for global injectivity:

$$F_1 = (f_1, g_1) = \left(2x - y + \frac{1}{27}(3x + y)^3,\ 3x - y + \frac{4}{45}(3x + y)^3\right).$$

For this polynomial map F_1, we observe that the degree of f_1 and g_1 is 3, $\det DF_1(x, y) = 1 + 2(3x + y)^2/3 > 0$, and the homogeneous polynomials $\overline{f}_1$ and $\overline{g}_1$ have in common the real linear factor $3x + y$. We recall that in (Braun and dos Santos Filho 2010) it is proved that all the polynomial maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ with one of the components with degree less than or equal to 3 whose Jacobian is nowhere zero, are injective, so F_1 is injective.

Now we shall show that there are polynomial maps satisfying the assumption of Theorem 1. We consider the following class of polynomial maps

$$F_2 = (f_2, g_2) = \big(-y + x^k - y^k, x + x^k + y^k\big) \text{ with } k \geq 1 \text{ odd.}$$

For these maps the degree of f_2 and g_2 is k, $\det DF_2(x, y) = 1 + k(x^{k-1} + y^{k-1}) + 2k^2 x^{k-1} y^{k-1} > 0$, and the homogeneous polynomials $\overline{f}_2$ and $\overline{g}_2$ have no common real linear factors, because the unique real linear factors of $\overline{f}_2$ and $\overline{g}_2$ are $x-y$ and $x+y$, respectively. Hence, by Theorem 1 it follows that the maps F_2 are injective.

In section 2 we summarize some results that we shall use in the proof of Theorem 1, which will be depicted in section 3. In section 4 we show that the Pinchuk counterexample to the real Jacobian conjecture does not satisfy the assumptions of Theorem 1.

2. Preliminary results

A singular point p of a differential system defined in $\mathbb{R}^2$ is a *center* if it has a neighborhood filled of periodic orbits with the unique exception of p. The *period annulus* of the center p is the maximal neighborhood $\mathcal{P}$ of p such that all the orbits contained in $\mathcal{P}$ are periodic except of course, p.

A center is *global* if its period annulus is the whole plane $\mathbb{R}^2$.

Let $\mathcal{X}$ be a planar polynomial vector field of degree n. The *Poincaré compactified vector field $p(\mathcal{X})$ corresponding to $\mathcal{X}$* is an analytic vector field induced on $\mathbb{S}^2$ as follows, see for more details (González 1969) or Chapter 5 of (Dumortier et al. 2006).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at the point y. Assume that $\mathcal{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 \equiv \mathbb{R}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the open northern hemisphere $\mathbb{H}^+$ and other in the open southern hemisphere $\mathbb{H}^-$. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1 = \mathbb{H}^+ \cup \mathbb{H}^-$ there are two symmetric copies of $\mathcal{X}$, one in $\mathbb{H}^+$ and other in $\mathbb{H}^-$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The projection of $\mathbb{H}^+$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

The singular points of $\mathcal{X}$ are called the *finite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. While the singular points of $p(\mathcal{X})$ contained in $\mathbb{S}^1$, i.e. at infinity, are called the *infinite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. It is known that the infinity singular points appear in pairs diametrally opposite.

Assume that the two components of the planar polynomial vector field $\mathcal{X}$ are the polynomials P and Q, such that n is the maximum of the degrees of P and Q. Denote as usual P_n and Q_n the homogeneous parts of degree n of P and Q, respectively. Then it is known that $p(\mathcal{X})$ has infinite singular points if and only if there exist real linear factors $ax + by$ dividing $yP_n - xQ_n$, see Chapter 5 of (Dumortier et al. 2006). In this situation, the endpoints of the straight line $ax + by = 0$ provide the infinite singular points.

Now we assume that Δ is an open subset of $\mathbb{R}^2$ and $\mathcal{X}$ is a vector field of class $\mathcal{C}^r$ with $r \geq 1$. For the basic definition of *ω–limit set* or *α–limit set* of an orbit, see for instance Chapter 1 of (Dumortier et al. 2006).

Theorem 2 (Poincaré–Bendixson Theorem)**.** *Let $\varphi(t) = \varphi(t, p)$ be an integral curve of $\mathcal{X}$ defined for all $t \geq 0$, such that $\varphi(0) = p$ and $\gamma_p^+ = \{\varphi(t) : t \geq 0\}$ is contained in a compact set $K \subset \Delta$. Assume that the vector field $\mathcal{X}$ has at most a finite number of singularities in K. Then one of the following statements holds.*

- (i) *If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbit.*
- (ii) *If $\omega(p)$ contains both regular and singular points, then $\omega(p)$ is a graphic, i.e. a set formed by orbits, each of them tending to one of the singular points contained in $\omega(p)$ as $t \to \pm\infty$.*
- (iii) *If $\omega(p)$ does not contain regular points, then $\omega(p)$ is a unique singular point.*

The Poincaré–Bendixson theorem can also be stated for α–limit sets.

The next result is the Poincaré–Hopf theorem for the Poincaré compactification of a polynomial vector field. For a proof, see Theorem 6.30 of (Dumortier et al. 2006).

Theorem 3. *Let $\mathcal{X}$ be a polynomial vector field. If $p(\mathcal{X})$ defined on the Poincaré sphere $\mathbb{S}^2$ has finitely many singular points, then the sum of their topological indices is two.*

The next result of Sabatini, see Theorem 2.3 of (Sabatini 1998), will play a main role in the proof of Theorem 1.

Theorem 4. *Let $F = (f, g)$ be a polynomial map with nonvanishing Jacobian determinant such that $F(0, 0) = (0, 0)$. Then the following properties are equivalent.*

- (1) *The origin is a global centre for the polynomial differential system $\dot{x} = -ff_y - gg_y$, $\dot{y} = ff_x + gg_x$.*
- (2) *F is a global diffeomorphism of the plane onto itself.*

3. Proof of Theorem 1

We denote $(a_1, a_2) = F(0,0)$ and consider the translation $A(x,y) = (x - a_1, y - a_2)$. Taking the map $G = A \circ F$, we observe that $G(0,0) = (0,0)$, $\det DG$ is nowhere zero, the degrees of the components of G are equal and the assumption of Theorem 1 is still true for G, because the higher degree terms of F and G coincide. Moreover, F is injective if and only if G is injective. In what follows we will assume $F = G$.

We consider now the function $H_F : \mathbb{R}^2 \to \mathbb{R}$ defined by $H_F(x,y) = \left(f(x,y)^2 + g(x,y)^2\right)/2$ and the Hamiltonian vector field associated to H_F, $\mathcal{X} = (P,Q)$, i.e

$$\begin{aligned} \dot{x} &= P = -f f_y - g g_y, \\ \dot{y} &= Q = f f_x + g g_x. \end{aligned} \tag{1}$$

It is clear that if (x_0, y_0) is such that $F(x_0, y_0) = (0,0)$, then (x_0, y_0) is a singular point of $\mathcal{X}$. Moreover, (x_0, y_0) is an isolated minimum of H_F and so it is a center of the Hamiltonian system (1) because near (x_0, y_0) the level curves of H_F are closed.

By Theorem 4, in order to prove Theorem 1 it is enough to prove that $(0,0)$ is a global center of the polynomial differential system (1).

We now consider the Poincaré compactification $p(\mathcal{X})$ of $\mathcal{X}$ defined in $\mathbb{S}^2$.

We claim that $p(\mathcal{X})$ does not have infinite singular points. Indeed, there exist singular points of $p(\mathcal{X})$ at infinity if and only if there exist linear factors dividing the homogeneous polynomial$yP_n - xQ_n$, where n is the maximum degree of P and Q. Let m be the degree of the polynomials f and g. It is clear that $n \leq 2m - 1$. Moreover, by the Euler's Theorem for homogeneous functions it follows that

$$-y(f_m f_{m_y} + g_m g_{m_y}) - x(f_m f_{m_x} + g_m g_{m_x}) = -m({f_m}^2 + {g_m}^2), \tag{2}$$

and so the homogeneous part of degree $2m-1$ of P or Q is not zero, proving $n = 2m-1$. The same calculation (2) also shows that a linear factor divides $yP_n - xQ_n$ if and only if it divides $f_m = \overline{f}$ and $g_m = \overline{g}$, which does not occur by assumption. So the claim is proved and it follows that $\mathbb{S}^1$ is a periodic orbit of $p(\mathcal{X})$.

We claim that system (1) *has no finite singular points, but the origin.* Indeed, $P(x,y) = Q(x,y) = 0$ is equivalent to

$$\begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives that $f(x,y) = g(x,y) = 0$, since $\det DF(x,y) \neq 0$. Thus all the finite singular points of $\mathcal{X}$ are zeros of F, and consequently there are just a finite number of them. Moreover, by the considerations on the function H_F above, all the finite singular points of $\mathcal{X}$ are centers, each of them producing two centers of $p(\mathcal{X})$ in $\mathbb{S}^2$ (one in $\mathbb{H}^+$ and one in $\mathbb{H}^-$). As there are no infinite singular points, it follows by Theorem 3 that the sum of the indices of the singular points of $p(\mathcal{X})$ is 2. Since each center has index 1, it follows that $p(\mathcal{X})$ has only two singular points, and thus $\mathcal{X}$ has only one singular point. This singular point is $(0,0)$. Hence, the claim is proven.

Now we shall prove that $(0,0)$ is a global center of $p(\mathcal{X})$. Then $(0,0)$ will be a global center of $\mathcal{X}$, and by Theorem 4 the proof of Theorem 1 will be finished.

From now on we will consider $p(\mathcal{X})$ the projection of the Poincaré compactification on $\mathbb{D}^2$. Since there are no finite singular points, except the origin, and there are no infinite singular points, the boundary of the period annulus $\mathcal{P}$ of the center $(0,0)$ is a periodic orbit that we call γ_c. If it is $\mathbb{S}^1$, we are done. If not, in a neighborhood of γ_c in the exterior of the period annulus we take the orbit γ_a through some point a in this region, and *we claim that γ_a has its ω–limit or its α–limit set equal to γ_c.* Indeed by the Poincaré–Bendixson Theorem (see Theorem 2), these ω– or α–limit sets are either a singular point, a graphic, or a periodic orbit. Since in $\mathbb{D}^2 \setminus \mathcal{P}$ there are no singular points of $p(\mathcal{X})$, such ω– and α–limit sets are periodic orbits. This implies that γ_c outside $\mathcal{P}$ is stable or unstable, i.e. the orbits near it outside $\mathcal{P}$ spiral in forward or backward time to it. This completes the proof of the claim.

Considering now the Poincaré map defined in a transversal section S through γ_c, we observe that it is the identity map in $S \cap \mathcal{P}$, and it is different from the identity in $S \cap (\mathbb{D}^2 \setminus \mathcal{P})$. But this is impossible, because the Poincaré map is an analytic function since the vector field $p(\mathcal{X})$ is analytic. Therefore, the center is global and this completes the proof of Theorem 1.

Remark 5. *Analyzing the proof of Theorem 1, it is tempting to think that it can also be done under the hypothesis degree of f greater than degree of g. The hypothesis* (2) *in such a version of Theorem 1 would be that there are no real linear factors dividing $\overline{f}$. The problem is that this assumption guarantees that f is not a submersion, a necessary condition for* $\det DF$ *to be nowhere zero. Indeed, if f is a submersion, then the vector field $\mathcal{Y} = (f_y, -f_x)$ has no finite singular points. Since the Poincaré compactification of $\mathcal{Y}$ defined in $\mathbb{S}^2$, an even dimensional sphere, is a smooth vector field by the Poincaré–Hopf Theorem (see for instance Theorem 6.30 of (Dumortier et al. 2006), it must have a* *singular point, which will be an infinite one. Then a real linear factor divides $y\overline{f}_y + x\overline{f}_x = m\overline{f}$, where m is the degree of f.*

4. Pinchuk counterexample

The above mentioned Pinchuk example is $F(x, y) = (p(x, y), q(x, y))$ with p and q as follows. Let $t = xy - 1$, $h = t(xt + 1)$, $f = (xt + 1)^2(t^2 + y)$ and define $p(x, y) = h + f$. Observe that p has degree 10. The polynomial $q(x, y)$ varies for different Pinchuk maps, but always has the form $q(x, y) = -t^2 - 6th(h+1) - u(f, h)$, where u is an auxiliary polynomial in f and h, chosen so that the Jacobian determinant of (p, q) is $t^2 + (t + f(13 + 15h))^2 + f^2$, which is strictly positive. This $u(f, h)$ is a solution of a differential equation.

In the original paper (Pinchuk 1994) it is suggested the following u:

$$u = -\frac{75}{4}f^4 + (-75h - 67)f^3 + \left(\frac{225}{2}h^2 - 201h - 91\right) f^2 - 12fh,$$

which gives q of degree 40, since the higher degree of q comes from the term with f^4 contained in u. So the map (p, q) does not satisfy the hypothesis of Theorem 1 that the degrees of p and q must be equal.

In (Campbell 2011) the polynomial u is taken

$$u = 170fh + 91h^2 + 195fh^2 + 69h^3 + 75fh^3 + 75h^4/4,$$

which gives q of degree 25, since the higher degree comes from the term with $75fh^3$ contained in u. So the degrees of p and q are also not equal.

We remark that 25 is the smallest degree that a component q in a Pinchuk map can have. For details see (Campbell 2011).

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Resumo

Seja $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ uma aplicação polinomial tal que det $DF(x)$ é diferente de zero para todos $x \in \mathbb{R}^2$. Assumimos que os graus de f e g são iguais. Denotamos por $\overline{f}$ e $\overline{g}$ as partes homogêneas de maior grau de f e g, respectivamente. Nesta nota, damos uma demonstração baseada

na teoria qualitativa de equações diferenciais do seguinte resultado: Se $\overline{f}$ e $\overline{g}$ não têm fatores lineares em comum, então F é injetora.

Palavras-chave: Conjectura Jacobiana real, injetividade global, centro, compactificação de Poincaré.

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[1] Departamento de Matemática, Universidade Federal de São Carlos Rod. Washington Luís, Km 235, C.P. 676, 13565–905 São Carlos, SP–Brasil

E-mail address: `franciscobraun@dm.ufscar.br`

[2] Departament de Matemàtiques Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`