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A new qualitative proof of a result on the real jacobian conjecture

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ABSTRACT

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that det $DF(x)$ is different from zero for all $x \in \mathbb{R}^2$. We assume that the degrees of f and g are equal. We denote by $\overline{f}$ and $\overline{g}$ the homogeneous part of higher degree of f and g, respectively. In this note we provide a proof relied on qualitative theory of differential equations of the following result: If $\overline{f}$ and $\overline{g}$ do not have real linear factors in common, then F is injective.

Key words: Real Jacobian conjecture, global injectivity, center, Poincaré compactification.

1- INTRODUCTION

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that det DF is nowhere zero. It is clear that F is a local diffeomorphism, but it is not always injective. There are very general well known conditions to ensure that F is a global diffeomorphism, for instance F is a global diffeomorphism if and only if it is proper (i.e. if inverse images of compact subsets are compact), or F is a diffeomorphism if and only if $\int_0^\infty \inf_{|x| \leq s} \|DF(x)^{-1}\|^{-1} ds = \infty$. These conditions are due to Banach-Mazur and Hadamard, respectively, and remain true in more general spaces, for details, see (Plastock 1974). Another condition, now specifically of $\mathbb{R}^2$ and ensuring just the injectivity of F, is the following sufficient condition: the real eigenvalues of $DF(x)$, for all $x \in \mathbb{R}^2$, are not contained in an interval of the form $(0, \varepsilon)$, for some $\varepsilon > 0$, see (Fernandes et al. 2004) and (Cobo et al. 2002).

Now, if F is a polynomial map, the statement that F is injective is known as the *real Jacobian conjecture*. This conjecture is false, since Pinchuk constructed, in (Pinchuk 1994), a non injective polynomial map with nonvanishing Jacobian determinant. So it is natural to ask for additional conditions in order for this conjecture to hold. In (Braun and dos Santos Filho 2010), for example, it is shown that it is enough to assume that the degree of f is less or equal to 3. On the other hand, if we assume that det $DF(x)$ = constant $\neq 0$, then to know if F is injective is an open problem largely known as the *Jacobian conjecture*, see (Bass et al. 1982) and (Van den Essen 2000) for details and for surveys on the Jacobian conjecture.

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