J. London Math. Soc. (2) 72 (2005) 689–716
doi:10.1112/S0024610705007015

HERMAN RINGS AND ARNOLD DISKS

XAVIER BUFF, NÚRIA FAGELLA, LUKAS GEYER
AND CHRISTIAN HENRIKSEN

Abstract

For $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$, let $f_{\lambda,a}$ be the rational map defined by $f_{\lambda,a}(z) = \lambda z^2(az+1)/(z+a)$. If $\alpha \in \mathbb{R}/\mathbb{Z}$ is a Brjuno number, we let $\mathcal{D}_\alpha$ be the set of parameters (λ, a) such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α (we consider that $(e^{2i\pi\alpha}, 0) \in \mathcal{D}_\alpha$). Results obtained by McMullen and Sullivan imply that, for any $g \in \mathcal{D}_\alpha$, the connected component of $\mathcal{D}_\alpha \cap (\mathbb{C}^* \times (\mathbb{C} \setminus \{0,1\}))$ that contains g is isomorphic to a punctured disk.

We show that there is a holomorphic injection $\mathcal{F}_\alpha : \mathbb{D} \longrightarrow \mathcal{D}_\alpha$ such that $\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0)$ and $\mathcal{F}'_\alpha(0) = (0, r_\alpha)$, where r_α is the conformal radius at 0 of the Siegel disk of the quadratic polynomial $z \longmapsto e^{2i\pi\alpha}z(1+z)$.

As a consequence, we show that for $a \in (0, 1/3)$, if $f_{\lambda,a}$ has a fixed Herman ring with rotation number α and if m_a is the modulus of the Herman ring, then, as $a \to 0$, we have $e^{\pi m_a} = (r_\alpha/a) + \mathcal{O}(a)$.

We finally explain how to adapt the results to the complex standard family $z \longmapsto \lambda z e^{(a/2)(z-1/z)}$.

1. *Introduction*

In this paper, we are mainly concerned with the dynamics of rational maps of the form

$$f_{\lambda,a}(z) = \lambda z^2 \frac{az+1}{z+a}, \qquad \lambda \in \mathbb{C}^*,\ a \in \mathbb{C}.$$

Note that $f_{\lambda,a}$ is conjugate to $f_{\lambda,-a}$ via the conjugacy $z \longmapsto -z$. If $\lambda \in S^1$ and $a = 0$, then the map $f_{\lambda,a}$ is the rotation $z \longmapsto \lambda z$. Observe that when a is real and $|\lambda| = 1$, the map $f_{\lambda,a}$ is a Blaschke fraction $z \longmapsto \lambda z^2(z+b)/(1+\bar{b}z)$ with $b = 1/a$. However, as opposed to families of Blaschke fractions which only depend $\mathbb{R}$-analytically on parameters, our family depends $\mathbb{C}$-analytically on the parameters λ and a and is, in some sense, the simplest one that exhibits families of Herman rings.

For all $\alpha \in \mathbb{R}/\mathbb{Z}$, we denote by $\mathcal{R}_\alpha$ the rigid rotation of the complex plane $\mathcal{R}_\alpha(z) = e^{2i\pi\alpha}z$. When $\lambda \in S^1$ and $a \in (-1/3, 1/3)$, the map $f_{\lambda,a}$ restricts to a diffeomorphism of S^1 which has a rotation number $\rho(\lambda, a) \in \mathbb{R}/\mathbb{Z}$. Given $a \in (-1/3, 1/3)$, the function $t \longmapsto \rho(e^{2i\pi t}, a)$ is continuous and weakly increasing [**15**]. Moreover, for each fixed $a \in (-1/3, 1/3)$ and for each irrational number α, there is a unique angle $t \in \mathbb{R}/\mathbb{Z}$ such that $\rho(e^{2i\pi t}, a) = \alpha$ (see for example [**6**]). By a theorem of Denjoy [**7**], when $\alpha = \rho(\lambda, a)$ is irrational, $f_{\lambda,a} : S^1 \longrightarrow S^1$ is topologically conjugate to the rotation $\mathcal{R}_\alpha : S^1 \longrightarrow S^1$.

Received 19 May 2004; revised 8 March 2005.

2000 *Mathematics Subject Classification* 37F10, 30D20.

The second author was partially supported by CIRIT grant 2001SGR-70 and MCyT grants BFM2002-01344 and BFM2003-9504. The third author was partially supported by the Alexander von Humboldt Foundation. The fourth author was supported by an SNF Steno stipend.