SECOND BOGOLUBOV'S THEOREM FOR LIPSCHITZ SYSTEMS AND BIFURCATIONS OF ASYMPTOTICALLY STABLE PERIODIC SOLUTIONS IN DIFFERENTIAL EQUATIONS WITH JUMPING NONLINEARITIES

A. BUICĂ

Department of Applied Mathematics, Babeş–Bolyai University, Cluj–Napoca, 400084, Romania
E-mail: abuica@math.ubbcluj.ro
www.math.ubbcluj.ro/~ abuica

J. LLIBRE

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, 08193, Spain
E-mail: jllibre@mat.uab.cat
www.gsd.uab.es/personal/jllibre

O. MAKARENKOV

Research Institute of Mathematics, Voronezh State University, Voronezh, 394006, Russia
E-mail: omakarenkov@math.vsu.ru

The goal of this paper is to study bifurcations of asymptotically stable 2π-periodic solutions in the forced asymmetric oscillator $\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^+ = \varepsilon\lambda \cos t$ by means of a Lipschitz generalization of the second Bogolubov's theorem due to the authors. The small parameter $\varepsilon > 0$ is introduced in such a way that any solution of the system corresponding to $\varepsilon = 0$ is 2π-periodic. We show that exactly one of these solutions whose amplitude is $2|\lambda|/\sqrt{a^2 + 4c^2}$ generates a branch of 2π-periodic solutions when $\varepsilon > 0$ increases. The solutions of this branch are asymptotically stable provided that $c > 0$.

Keywords: Differential equations with jumping nonlinearities; Asymptotically stable periodic solutions; Bifurcation.

1. Introduction

The differential equation for the coordinate u of the mass attached via nonlinear spring to an immovable beam drawn at Fig. 1 is written down as follows

$$m\ddot{u} + c\dot{u} + k_1 u + k_2 u^+ = f(t), \tag{1}$$

where f is a force applied to the mass in the vertical direction.

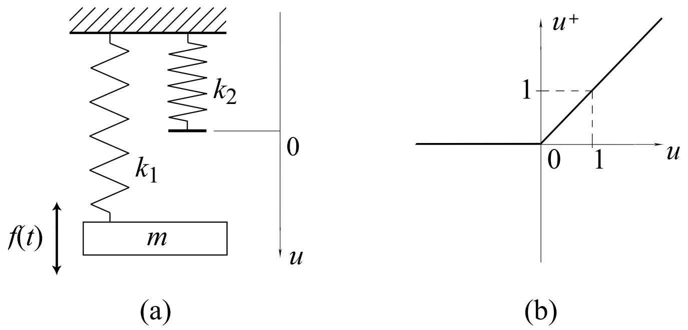

Fig. 1. (a) A driven mass attached to an immovable beam via a spring with piecewise linear stiffness, see e.g. Ref. 5, (b) the jumping nonlinearity $u \mapsto u^+$.

A complete theory for studying bifurcation of asymptotically stable periodic solutions in the following form of equation (1)

$$m\ddot{u} + \varepsilon c_\varepsilon \dot{u} + k_1 u + k_2 u^+ = \varepsilon f(t), \tag{2}$$

has been developed by Glover, Lazer and McKenna in their pioneer work Ref. 4 provided that $c_\varepsilon \to 0$ as $\varepsilon \to 0$. They showed that if the unperturbed system $m\ddot{u} + k_1 u + k_2 u^+ = 0$ has a T-periodic orbit u_0, the function f is T-periodic and for some $\alpha \in [0,T]$ we have that $\int_0^T \dot{u}_0(\tau) f(\tau - \alpha) d\tau = 0$ and $d = \int_0^T \ddot{u}_0(\tau) f(\tau - \alpha) d\tau > 0$, then the T-periodic solution u_0 persists and becomes asymptotically stable as $\varepsilon > 0$ increases. In other words the authors of Ref. 4 showed that the conclusion of the second Bogolubov's theorem[1] is valid also for equation (2), even this is not C^1.

Second Bogolubov's theorem. *Consider the perturbed system*

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{3}$$

where $g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ *is* T*-periodic in the first variable. If* $v_0 \in \mathbb{R}^n$ *is a zero of the bifurcation function*

$$g_0(v) = \int_0^T g(\tau, v, 0)\, d\tau \tag{4}$$

and $\det (g_0)'(v_0) \neq 0$*, then for any* $\varepsilon > 0$ *sufficiently small system (3) has a unique* T*-periodic solution* x_ε *such that* $x_\varepsilon(0) \to v_0$ *as* $\varepsilon \to 0$*. If, in* *addition, all the eigenvalues of the matrix* $(g_0)'(v_0)$ *have negative real part, then* x_ε *is asymptotically stable.*

We note that equation (2), after a convenient change of variables can be written into the standard form (4) of averaging theory (see a similar example in Section 3).

In our talk at the Equadiff 2007 we presented a general class of Lipschitz systems (which includes, in particular, equation (2)) for which the conclusion of the second Bogolubov's theorem holds. This result is formulated in detail in the next section of the paper. Though it allows to treat more complex equations than (2), we study in Section 3 the following particular case:

$$\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^+ = \varepsilon\lambda\cos t\,. \tag{5}$$

On one hand, this equation cannot be formally studied by the theorems from Ref. 4, and on the other hand, applying our result (see Theorem 2.1 below) we obtain explicit conditions for the coefficients c, a and λ that guarantee the bifurcation of a branch of periodic solutions. In this way we answer to the question of Jean Mawhin about the values of the parameters for which the bifurcation occurs.

Using other conditions for the parameters of (1) Lazer-McKenna[6] and Fabry[3] also studied the existence and stability of 2π-periodic solutions of (1). We mention that in Ref. 6 it is assumed that the amplitude of the forcing term f is sufficiently large, while in Ref. 3 the authors address periodic solutions whose amplitude goes to $+\infty$ as a suitable small parameter $\varepsilon > 0$

2. Lipschitz generalization of the second Bogolubov's theorem.

Throughout the paper $\Omega \subset \mathbb{R}^k$ is some open set. For any $\delta > 0$ we denote $B_\delta(v_0) = \left\{v \in \mathbb{R}^k \ : \ \|v - v_0\| \le \delta\right\}$. For any set $M \subset [0,T]$ measurable in the sense of Lebesgue we denote by $\mathrm{mes}(M)$ the Lebesgue measure of M. We have the following main result proved in Ref. 2.

Theorem 2.1. *Let* $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ *and* $v_0 \in \Omega$. *Let* g_0 *be the averaging function given by (4) and consider* $v_0 \in \Omega$ *such that* $g_0(v_0) = 0$. *Assume that:*

(i) For some $L > 0$ *we have that* $\|g(t,v_1,\varepsilon) - g(t,v_2,\varepsilon)\| \le L\,\|v_1 - v_2\|$ *for any* $t \in [0,T]$, $v_1, v_2 \in \Omega$, $\varepsilon \in [0,1]$ *;*

(ii) given any $\gamma > 0$ *there exist* $\delta > 0$ *and* $M \subset [0,T]$ *measurable in the sense of Lebesgue with* $\mathrm{mes}(M) < \gamma$ *such that for every* $v \in B_\delta(v_0)$,

$t \in [0,T] \setminus M$ and $\varepsilon \in [0,\delta]$ we have that $g(t,\cdot,\varepsilon)$ is differentiable at v and $\|g'_v(t,v,\varepsilon) - g'_v(t,v_0,0)\| \le \gamma$;

(iii) g_0 is continuously differentiable in a neighborhood of v_0 and the real parts of all the eigenvalues of $(g_0)'(v_0)$ are negative.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0,\delta_1]$, system (3) has exactly one T–periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

3. Bifurcations of asymptotically stable periodic solutions in differential equations with jumping nonlinearities

In this section we apply Theorem 2.1 to studying the bifurcation of asymptotically stable 2π-periodic solutions in equation (5).
Some function u is a solution of (5) if and only if $(z_1, z_2) = (u, \dot{u})$ is a solution of the system

$$\begin{aligned} \dot{z}_1 &= z_2\,, \\ \dot{z}_2 &= -z_1 + \varepsilon[-az_1^+ - cz_2 + \lambda\cos t]\,. \end{aligned} \tag{6}$$

After the change of variables

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

system (6) takes the form

$$\begin{aligned} \dot{x}_1 &= \varepsilon \sin t\,[a(x_1\cos t + x_2 \sin t)^+ + c(-x_1\sin t + x_2\cos t) - \lambda\cos t]\,, \\ \dot{x}_2 &= \varepsilon \cos t\,[-a(x_1\cos t + x_2\sin t)^+ + c(x_1\sin t - x_2\cos t) + \lambda\cos t]\,. \end{aligned} \tag{7}$$

The corresponding averaged function g_0, calculated according to the formula (4), is

$$g_0(x_1,x_2) = \begin{pmatrix} -\pi c & \pi a/2 \\ -\pi a/2 & -\pi c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi\lambda \end{pmatrix}.$$

It can be easily checked that the unique zero of g_0 is

$$\left(\frac{2a\lambda}{a^2+4c^2}, \frac{4c\lambda}{a^2+4c^2} \right)$$

and

$$\text{the eigenvalues of } (g_0)' \text{ are } -\pi c \pm i\pi a\,. \tag{8}$$

The amplitude of this zero is

$$A = \frac{2|\lambda|}{\sqrt{a^2+4c^2}}.$$

To apply Theorem 2.1 it remains to prove the following proposition.

Proposition 3.1. *Let $v_0 \in \mathbb{R}^2 \setminus \{0\}$. Then the right hand side of (7) satisfies (ii) for any $c, a, \lambda \in \mathbb{R}$.*

Proof. Let $[v]_i$ be the i–th component of the vector $v \in \mathbb{R}^2$. Let $g(t,v) = ([v]_1 \cos t + [v]_2 \sin t)^+$ and notice that it is enough to prove that $g : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (ii). In the case that $[v_0]_2 \neq 0$, denote $\theta(v) = \arctan(-[v]_1/[v]_2)$, while when $[v_0]_2 = 0$, denote

$$\theta(v) = \begin{cases} \arctan(-[v]_1/[v]_2) & \text{if } \ [v_0]_1[v]_2 < 0, \\ \pi/2 & \text{if } \ v = v_0, \\ \arctan(-[v]_1/[v]_2) + \pi & \text{if } \ [v_0]_1[v]_2 > 0. \end{cases}$$

In any case notice that the function $v \mapsto \theta(v)$ is continuous in every sufficiently small neighborhood of v_0. Fix $\gamma > 0$. Let M be the union of the interval M_1 centered in $\theta(v_0)$ (when $\theta(v_0) < 0$, take $\theta(v_0) + 2\pi$ instead of $\theta(v_0)$) and of the interval M_2 centered in $\theta(v_0) + \pi$, each of length $\gamma/2$. Take $\delta > 0$ such that $\theta(v) \in M_1$ for all $v \in B_\delta(v_0)$. Of course, also $\theta(v) + \pi \in M_2$ for all $v \in B_\delta(v_0)$. This implies that for fixed $t \in [0, 2\pi] \setminus M$, $[v]_1 \cos t + [v]_2 \sin t$ has constant sign for all $v \in B_\delta(v_0)$. Therefore, $g(t, \cdot)$ is differentiable and $g'_v(t,v) = g'_v(t, v_0)$ for all $v \in B_\delta(v_0)$. Hence (ii) is fulfilled. □

The main result of this section can be now summarized as follows.

Theorem 3.1. *Assume that $c > 0$ and $A = 2|\lambda|/\sqrt{a^2 + 4c^2} \neq 0$ and take an arbitrary $R > 0$. Then for each $\varepsilon > 0$ sufficiently small, equation (5) has an asymptotically stable 2π-periodic solution whose amplitude goes to A as $\varepsilon \to 0$. Moreover, there are no other 2π-periodic solutions with amplitudes in the interval $[1/R, R]$.*

Proof. The hypotheses of Theorem 2.1 holds because (i) is immediate, (ii) is proved in Proposition 3.1 and (iii) follows by (8). Hence, the existence of a unique branch of asymptotically stable 2π-periodic solution whose amplitude goes to A as $\varepsilon \to 0$ is proved. The absence of other 2π-periodic solutions follows from a Mawhin's result[7] and the fact that the averaged function g_0 has a unique zero. Indeed, the result of Mawhin assures that the initial value of a 2π-periodic solution of (5) must converges to a zero of the averaged function g_0 as $\varepsilon \to 0$. □

By Theorem 3.1 the curves of dependence of the amplitude of asymptotically stable 2π-periodic oscillations in (5) upon the parameters are drawn in Fig. 2. Particularly one can see that this amplitude tends to $+\infty$ as $\sqrt{a^2+4c^2} \to 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$ is fixed.

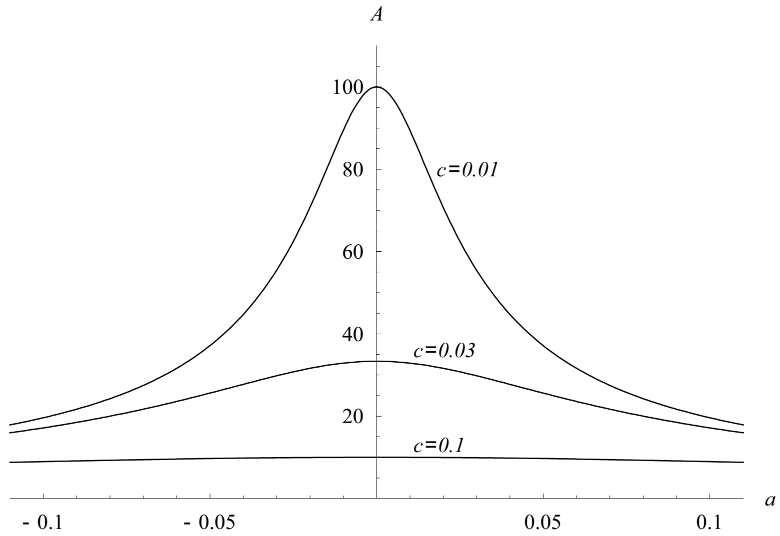

Fig. 2. The curves of dependence of the amplitude of asymptotically stable 2π-periodic oscillations in (5) upon the parameter $a \in \mathbb{R}$ constructed for $\lambda = 1$ and distinct c's.

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