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MATHEMATICS

On Yu.A. Mitropol'skii's Theorem on Periodic Solutions of Systems of Nonlinear Differential Equations with Nondifferentiable Right-Hand Sides[1]

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This papers studies the existence, uniqueness and asymptotic stability of T-periodic solutions for the system

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{1}$$

where $\varepsilon > 0$ is a small parameter and the function $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$ is T-periodic in the first variable and locally Lipschitz with respect to the second variable. As usual, a key role is played by the averaged function

$$g_0(v) = \int_0^T g(\tau, v, 0)d\tau, \tag{2}$$

and we are interested in T-periodic solutions of system (1) that start near $v_0 \in (g_0)^{-1}(0)$.

In the case where g is continuously differentiable, i.e., is of class $C^1(R \times \mathbb{R}^k, \mathbb{R}^k)$, the periodic case of Bogolyubov's second theorem ([2], Chapter 1, Section 5, Theorem II) asserts that, *for small* $\varepsilon > 0$, *the condition* $\det(g_0)'(v_0) \neq 0$ *ensures the existence and uniqueness of T-periodic solution of system* (1) *in a neighborhood of* v_0, *and if all eigenvalues of the matrix* $(g_0)'(v_0)$ *have negative real part*, *then these solutions are asymptotically stable*.

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In [9], Mitropol'skii noticed that some applications require the generalization of Bogolyubov's second theorem for perturbations g satisfying only the Lipschitz conditions. For such functions g, assuming that $g_0 \in C^3(\mathbb{R}^k, \mathbb{R}^k)$ and all eigenvalues of the matrix $(g_0)'(v_0)$ have negative real part, Mitropol'skii suggested an analogue of Bogolyubov's second theorem; namely, he proved the existence and uniqueness of a T-periodic solution of system (1) in a neighborhood of v_0. Later, Mitropol'skii's existence theorem was strongly generalized (see [11, 8]), and analogues of his uniqueness result for equations with monotone nonlinearities were obtained (see [10, 12]). Nevertheless, the asymptotic stability conclusion of Bogolyubov's second theorem in Mitropol'skii's setting (for Lipschitz g) has not been generalized for a long time. It was generalized only recently by Buica and Daniilidis in [3] for a class of functions $v \mapsto g(t, v, 0)$ differentiable at v_0 for almost all $t \in [0, T]$, but it is assumed in [3] that the eigenvectors of the matrix $(g_0)'(v_0)$ are orthogonal and the function g has a continuous Clarke differential, which is not easy to check in applications.

In the next section of this paper, assuming that g is piecewise differentiable in the second variable, we show that Mitropol'skii's conditions imply not only the uniqueness but also the asymptotic stability of a T-periodic solution of system (1) in a neighborhood of v_0 (see Theorem 2 below). In other words, we prove that Bogolyubov's theorem stated above is valid even for nondifferentiable functions g. Theorem 2 follows from our even more general Theorem 1, which does not assume neither g nor g_0 to be differentiable.

1. Throughout the paper, $\Omega \subset \mathbb{R}^k$ is an open set and, for any $v \in \mathbb{R}^k$, its δ-neighborhood is defined by $B_\delta(v_0) = \{v \in \mathbb{R}^k: \|v - v_0\| \leq \delta\}$. The following theorem is the main result of this paper.