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ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR NONSMOOTH DIFFERENTIAL EQUATIONS WITH APPLICATION TO THE NONSMOOTH VAN DER POL OSCILLATOR*

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Abstract. In this paper we study the existence, uniqueness, and asymptotic stability of the periodic solutions of the Lipschitz system $\dot{x} = \varepsilon g(t, x, \varepsilon)$, where $\varepsilon > 0$ is small. Our results extend the classical second Bogoliubov theorem for the existence of stable periodic solutions to nonsmooth differential systems. As an application we prove the existence of asymptotically stable 2π-periodic solutions of the nonsmooth van der Pol oscillator $\ddot{u} + \varepsilon (|u| - 1) \dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t$. Moreover, we construct the so-called resonance curves that describe the dependence of the amplitude of these solutions as a function of the parameters a and λ. Finally we compare such curves with the resonance curves of the classical van der Pol oscillator $\ddot{u} + \varepsilon \left(u^2 - 1\right) \dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t$.

Key words. periodic solution, asymptotic stability, averaging theory, nonsmooth differential system, nonsmooth van der Pol oscillator

AMS subject classifications. 34C29, 34C25, 47H11

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1. Introduction.

In this paper we study the existence, uniqueness, and asymptotic stability of the T-periodic solutions of the system

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and the function $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$ is T-periodic in the first variable and locally Lipschitz with respect to the second. For this class of differential systems, the study of the T-periodic solutions can be made using the averaging function

$$g_0(v) = \int_0^T g(\tau, v, 0) d\tau \tag{1.2}$$

and looking for the periodic solutions that starts near some $v_0 \in g_0^{-1}(0)$.

In the case that g is of class C^1, we recall the stable periodic case of the second Bogoliubov's theorem [6, Chap. 1, section 5, Theorem II] which states *If* $\det (g_0)'(v_0) \neq 0$ *and* $\varepsilon > 0$ *is sufficiently small, then system* (1.1) *has a unique T-periodic solution in a neighborhood of v_0. Moreover, if all the eigenvalues of the Jacobian matrix*

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