HOPF BIFURCATION IN THE FULL REPRESSILATOR EQUATIONS

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Abstract. In this paper we prove that the full repressilator equations, in dimension six undergo a supercritical Hopf bifurcation.

1. Introduction

Oscillatory networks are a particular kind of regulatory molecular networks, i.e., collections of interacting molecules in a cell. The regulatory oscillators can be used to study abnormalities of a process in the cell, from sleep disorders to cancer. So, they attract significant attention among biologists and biophysicists. There are many implementations of artificial oscillatory networks (see, e.g., [1, 2, 6, 7, 8, 10, 11]). One of them is the repressilator [5]. Its genetic implementation uses three proteins that cyclically repress the synthesis of one another. The following system of differential equations describes the behavior of the repressilator:

$$
\begin{aligned}
\dot{m}_1 &= -m_1 + \frac{\alpha}{1+v^n} + \alpha_0,\\
\dot{m}_2 &= -m_2 + \frac{\alpha}{1+w^n} + \alpha_0,\\
\dot{m}_3 &= -m_3 + \frac{\alpha}{1+u^n} + \alpha_0,\\
\dot{u} &= -\beta(u - m_1),\\
\dot{v} &= -\beta(v - m_2),\\
\dot{w} &= -\beta(w - m_3).
\end{aligned}
\tag{1}
$$

Here u, v and w are proportional to the protein concentration, while m_i are proportional to the concentration of mRNA corresponding to those proteins. The nonlinear function $f(x) = \dfrac{\alpha}{1+x^n}$ reflects synthesis of the mRNAs from the DNA controlled by regulatory elements. The parameter α_0 represents uncontrolled part of the mRNA synthesis, and it is usually small. The

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explicit inclusion of the mRNA concentration variables into the model is given by β. Given that in general $\beta \ll 1$ and α_0 is very small, we consider $\alpha_0 = \varepsilon a$ and $\beta = \varepsilon b$, where a and b are positive constants and $\varepsilon > 0$ is sufficiently small. So, system (1) becomes

$$
\begin{aligned}
\dot{m}_1 &= -m_1 + \frac{\alpha}{1+v^n} + \varepsilon a,\\
\dot{m}_2 &= -m_2 + \frac{\alpha}{1+w^n} + \varepsilon a,\\
\dot{m}_3 &= -m_3 + \frac{\alpha}{1+u^n} + \varepsilon a,\\
\dot{u} &= -\varepsilon b(u-m_1),\\
\dot{v} &= -\varepsilon b(v-m_2),\\
\dot{w} &= -\varepsilon b(w-m_3).
\end{aligned}
\tag{2}
$$

In the papers [3, 4] the authors consider a reduced system of dimension three. The reduction assumes that the three excluded variables, i.e. m_i, evolve an order of magnitude faster than the other three. In [3] the authors prove that in the reduced system exhibits a supercritical Hopf bifurcation. The existence of a Hopf bifurcation in the reduced system does not imply that the full system, in dimension six, also has a supercritical Hopf bifurcation. It just gives an indication about its existence. Here in this work we consider the full system in dimension six and we extend the results of [3, 4] on the supercritical Hopf bifurcation to the 6−dimensional differential system (2). Our main result is the following one.

Theorem 1. *Let $n > 2$ be an integer. The point $(r_0, r_0, r_0, r_0, r_0, r_0)$ with*

$$
r_0 = \sqrt[n]{\frac{2}{n-2}} + \varepsilon \frac{a - 3b\sqrt[n]{\dfrac{2}{n-2}}}{n-2} + \mathcal{O}(\varepsilon^2),
$$

is an equilibrium of the differential system (2) *with*

$$
\alpha = \alpha_{bif} = \frac{n}{n-2}\sqrt[n]{\frac{2}{n-2}} + \varepsilon\frac{n}{(n-2)^2}\left(-a(n-5) - 9b\sqrt[n]{\frac{2}{n-2}}\right) + \mathcal{O}(\varepsilon^2).
$$

The eigenvalues of the linear part of system (2) *at this equilibrium are* $\{\pm\varepsilon\sqrt{3}bi+\mathcal{O}(\varepsilon^2), -1+\varepsilon(-1\pm\sqrt{3}i)b+\mathcal{O}(\varepsilon^2), -\varepsilon 3b+\mathcal{O}(\varepsilon^2), -1+\varepsilon 2b+\mathcal{O}(\varepsilon^2)\}$. *Moreover, there is a single supercritical Hopf bifurcation at $\alpha = \alpha_{bif}$ and there exists a small $\varepsilon_0 > 0$ such that for $\alpha_{bif} < \alpha < \alpha_{bif} + \varepsilon_0$ the system* (2) *has a stable limit cycle.*

2. Proof of Theorem 1

It is clear that system (2) has the equilibrium $p_0 = (r_0, r_0, r_0, r_0, r_0, r_0)$ where r_0 is solution of the equation

$$\frac{\alpha}{1+r^n} = r - \varepsilon a. \tag{3}$$

From (3) we have that $\alpha = (r_0 - \varepsilon a)(1 + r_0^n)$. So, substituting α in the linear part of system (2) at the equilibrium p_0 we get

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & \Delta & 0 \\ 0 & -1 & 0 & 0 & 0 & \Delta \\ 0 & 0 & -1 & \Delta & 0 & 0 \\ \varepsilon b & 0 & 0 & -\varepsilon b & 0 & 0 \\ 0 & \varepsilon b & 0 & 0 & -\varepsilon b & 0 \\ 0 & 0 & \varepsilon b & 0 & 0 & -\varepsilon b \end{pmatrix},$$

where $\Delta = -\dfrac{n r_0^{-1+n}(r_0 - \varepsilon a)}{1 + r_0^n}$. The eigenvalues of M are

$$\varepsilon\frac{(-2 + (n - 2 \pm i\sqrt{3}n)r_0^n)b}{2(1 + r_0^n)} + \mathcal{O}(\varepsilon^2), \quad -1 + \varepsilon\frac{(-1 \pm i\sqrt{3})n r_0^n b}{2(1 + r_0^n)} + \mathcal{O}(\varepsilon^2),$$

$$-1 + \varepsilon\frac{n r_0^n b}{1 + r_0^n} + \mathcal{O}(\varepsilon^2), \qquad -\varepsilon\frac{(1 + (1 + n)r_0^n)\, b}{1 + r_0^n} + \mathcal{O}(\varepsilon^2).$$

We impose that the real part of the eigenvalues $\varepsilon\dfrac{(-2 + (n - 2 \pm i\sqrt{3}n)r_0^n)b}{2(1 + r_0^n)} + \mathcal{O}(\varepsilon^2)$ is zero and we obtain

$$r_0 = \sqrt[n]{\frac{2}{n-2}} + \varepsilon\frac{a - 3b\sqrt[n]{\dfrac{2}{n-2}}}{n-2} + \mathcal{O}(\varepsilon^2). \tag{4}$$

Substituting (4) in (3) we get

$$\alpha_{bif} = \frac{n}{n-2}\sqrt[n]{\frac{2}{n-2}} - \varepsilon\frac{n}{(n-2)^2}\left(a(n-5) + 9b\sqrt[n]{\frac{2}{n-2}}\right) + \mathcal{O}(\varepsilon^2). \tag{5}$$

Substituting (4) in M and computing the eigenvalues we obtain $\pm\varepsilon\sqrt{3}bi + \mathcal{O}(\varepsilon^2), -1 + \varepsilon(-1 \pm \sqrt{3}i)b + \mathcal{O}(\varepsilon^2), -\varepsilon 3b + \mathcal{O}(\varepsilon^2), -1 + \varepsilon 2b + \mathcal{O}(\varepsilon^2)$.

The linearization of (2) at p_0 has a pair of conjugate purely imaginary eigenvalues and the other four eigenvalues have negative real part. This is the setting for a Hopf bifurcation. We can expect to see a small-amplitude limit cycle branching from the fixed point p_0. It remains to compute the first Lyapunov coefficient $\ell_1(p_0)$ of (2) near p_0. When $\ell_1(p_0) < 0$ the point p_0 is a weak focus of system (2) restricted to the central manifold of p_0 and the limit cycle that emerges from p_0 is stable. In this case we say that the Hopf bifurcation is supercritical. When $\ell_1(p_0) > 0$ the point p_0 is also a weak focus of system (2) restricted to the central manifold of p_0 but the limit cycle that borns from p_0 is unstable. In this second case we say that the Hopf bifurcation is subcritical.

Here we use the following result presented on page 180 of the book [9] for computing $\ell_1(p_0)$.

Lemma 2. *Let $\dot{x} = F(x)$ be a differential system having p_0 as an equilibrium point. Consider the third order Taylor approximation of F around p_0 given by $F(x) = Ax + \frac{1}{2!}B(x,x) + \frac{1}{3!}C(x,x,x) + \mathcal{O}(|x|^4)$. Assume that A has a pair of purely imaginary eigenvalues $\pm\lambda i$. Let q be the eigenvector of A corresponding to the eigenvalue λi, normalized so that $\overline{q}\cdot q = 1$, where $\overline{q}$ is the conjugate vector of q. Let p be the adjoint eigenvector such that $A^T p = -\lambda i p$ and $\overline{p}\cdot q = 1$. If I denotes the 6×6 identity matrix, then*

$$\begin{aligned}\ell_1(p_0) = \frac{1}{2\lambda} Re(&\overline{p}\cdot C(q,q,\overline{q}) - 2\overline{p}\cdot B(q, A^{-1}B(q,\overline{q}))\\ &+\overline{p}\cdot B(\overline{q}, (2\lambda i I - A)^{-1}B(q,q))).\end{aligned}$$

In our case the linear part of system (2) at the equilibrium p_0 is

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & -2+\varepsilon\sigma & 0\\ 0 & -1 & 0 & 0 & 0 & -2+\varepsilon\sigma\\ 0 & 0 & -1 & -2+\varepsilon\sigma & 0 & 0\\ b\varepsilon & 0 & 0 & -b\varepsilon & 0 & 0\\ 0 & b\varepsilon & 0 & 0 & -b\varepsilon & 0\\ 0 & 0 & b\varepsilon & 0 & 0 & -b\varepsilon \end{pmatrix} + \mathcal{O}(\varepsilon^2),$$

where

$$\sigma = 6b + \frac{1}{(n-2)^2}\left(b^2(51-39n) + ba12\sqrt[n]{\frac{n-2}{2}} - a^2\sqrt[n]{\left(\frac{n-2}{2}\right)^2}(n-1)\right).$$

We have that A has an eigenvalue $\varepsilon\sqrt{3}bi + \mathcal{O}(\varepsilon^2)$. Now we compute the bi- and tri-linear functions B and C. Considering the vector field $(f_1, f_2, f_3, f_4, f_5, f_6)$ associated to the differential system (2) we observe that all second and third derivatives vanishes except $\dfrac{\partial^2 f_1}{\partial v^2}$, $\dfrac{\partial^2 f_2}{\partial w^2}$, $\dfrac{\partial^2 f_3}{\partial u^2}$, $\dfrac{\partial^3 f_1}{\partial v^3}$, $\dfrac{\partial^3 f_2}{\partial w^3}$ and $\dfrac{\partial^3 f_3}{\partial u^3}$. Computing these derivatives, taking into account (4) and (5) we get that

$$\frac{\partial^2 f_1}{\partial v^2}(p_0) = \frac{\partial^2 f_2}{\partial w^2}(p_0) = \frac{\partial^2 f_3}{\partial u^2}(p_0) = \gamma,$$

$$\frac{\partial^3 f_1}{\partial v^3}(p_0) = \frac{\partial^3 f_2}{\partial w^3}(p_0) = \frac{\partial^3 f_3}{\partial u^3}(p_0) = \delta,$$

where

$$\begin{aligned}\gamma = &\quad -2(n-5)\sqrt[n]{\frac{n-2}{2}} \\ &+\varepsilon\sqrt[n]{\frac{2^{n-2}}{(n-2)^{n-2}}}\left(a(5n-13)+3b\sqrt[n]{\frac{2}{n-2}}(n^2-12n+23)\right)+\mathcal{O}(\varepsilon^2),\end{aligned}$$

and

$$\begin{aligned}\delta = &\quad -2\sqrt[n]{\left(\frac{n-2}{2}\right)^2}(n^2-15n+38) \\ &+\varepsilon\frac{2}{n-2}\sqrt[n]{\left(\frac{n-2}{2}\right)^3}\Big(2a(7n^2-57n+98) \\ &+3b\sqrt[n]{\frac{2}{n-2}}(n^3-31n^2+182n-272)\Big)+\mathcal{O}(\varepsilon^2).\end{aligned}$$

So, the bilinear function B is given by

$$B((x_1,y_1,z_1,u_1,v_1,w_1),(x_2,y_2,z_2,u_2,v_2,w_2)) = (\gamma v_1 v_2, \gamma w_1 w_2, \gamma u_1 u_2, 0,0,0),$$

and the tri-linear function C is given by the expression

$$\begin{gathered}C((x_1,y_1,z_1,u_1,v_1,w_1),(x_2,y_2,z_2,u_2,v_2,w_2),(x_3,y_3,z_3,u_3,v_3,w_3)) = \\ (\delta v_1 v_2 v_3, \delta w_1 w_2 w_3, \delta u_1 u_2 u_3, 0,0,0).\end{gathered}$$

Computing the normalized eigenvector q of A, associated to the eigenvalue $\varepsilon\sqrt{3}bi + \mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned} q = \; & \Big(\frac{1-\sqrt{3}i}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{1+\sqrt{3}i}{\sqrt{15}}, \frac{-1-\sqrt{3}i}{2\sqrt{15}}, \frac{-1+\sqrt{3}i}{2\sqrt{15}}, \frac{1}{\sqrt{15}}\Big) \\ & +\varepsilon\Big(\frac{-30(\sqrt{3}+i)b-(\sqrt{3}-3i)\sigma}{30\sqrt{5}}, \frac{60bi+2\sqrt{3}\sigma}{30\sqrt{5}}, \frac{30(\sqrt{3}-i)b-(\sqrt{3}+3i)\sigma}{30\sqrt{5}}, \\ & -\frac{(2\sqrt{3}+6i)\sigma}{30\sqrt{5}}, -\frac{(2\sqrt{3}-6i)\sigma}{30\sqrt{5}}, \frac{4\sqrt{3}\sigma}{30\sqrt{5}}\Big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The normalized adjoint eigenvector of the transpose matrix A with the eigenvalue $-\varepsilon\sqrt{3}bi$ is

$$\begin{aligned} p = \; & \Big(0, 0, 0, \frac{-\sqrt{15}-3\sqrt{5}i}{6}, \frac{-\sqrt{15}+3\sqrt{5}i}{6}, \frac{\sqrt{15}}{3}\Big) \\ & +\varepsilon\Big(\frac{(-5-5\sqrt{3}i)b}{2\sqrt{15}}, \frac{(-5+5\sqrt{3}i)b}{2\sqrt{15}}, \frac{5b}{\sqrt{15}}, \frac{10b+(1+\sqrt{3}i)\sigma}{\sqrt{15}}, \\ & \frac{-5(1+\sqrt{3}i)b+(1-\sqrt{3}i)\sigma}{\sqrt{15}}, \frac{-5(1-\sqrt{3}i)b-2\sigma}{\sqrt{15}}\Big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

According to Lemma 2, in order to compute $\ell_1(p_0)$, we need to compute first A^{-1} and $(2\sqrt{3}b\varepsilon iI - A)^{-1}$. We have that $A^{-1} = \dfrac{1}{\varepsilon}A_{-1} + A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$,

$$A_{-1} = \frac{1}{9b}\begin{pmatrix} 0 & 0 & 0 & 8 & 2 & -4 \\ 0 & 0 & 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 2 & -4 & 8 \\ 0 & 0 & 0 & -1 & 2 & -4 \\ 0 & 0 & 0 & -4 & -1 & 2 \\ 0 & 0 & 0 & 2 & -4 & -1 \end{pmatrix},$$

$$A_0 = \frac{1}{27b}\begin{pmatrix} -3b & 6b & -12b & -4\sigma & 5\sigma & -4\sigma \\ -12b & -3b & 6b & -4\sigma & -4\sigma & 5\sigma \\ 6b & -12b & -3b & 5\sigma & -4\sigma & -4\sigma \\ -3b & 6b & -12b & -4\sigma & 5\sigma & -4\sigma \\ -12b & -3b & 6b & -4\sigma & -4\sigma & 5\sigma \\ 6b & -12b & -3b & 5\sigma & -4\sigma & -4\sigma \end{pmatrix} \quad \text{and}$$

$$A_1 = \frac{\sigma}{81b}\begin{pmatrix} -12b & 15b & -12b & -10\sigma & 8\sigma & -\sigma \\ -12b & -12b & 15b & -\sigma & -10\sigma & 8\sigma \\ 15b & -12b & -12b & 8\sigma & -\sigma & -10\sigma \\ -12b & 15b & -12b & -10\sigma & 8\sigma & -\sigma \\ -12b & -12b & 15b & -\sigma & -10\sigma & 8\sigma \\ 15b & -12b & -12b & 8\sigma & -\sigma & -10\sigma \end{pmatrix}.$$

and the expression of $(2\sqrt{3}b\varepsilon iI - A)^{-1}$ is very large and we present it in the appendix.

The first, second and third terms of $\ell_1(p_0)$ given in Lemma 2 respectively are

$$Re(\overline{p} \cdot C(q, q, \overline{q})) = \frac{1}{15}\sqrt[n]{\left(\frac{n-2}{2}\right)^2}(n^2 - 15n + 38)b\varepsilon + \mathcal{O}(\varepsilon^2),$$

$$Re(-2\overline{p} \cdot B(q, A^{-1}B(q, \overline{q}))) = -\frac{4}{45}\sqrt[n]{\left(\frac{n-2}{2}\right)^2}(n^2 - 10n + 25)b\varepsilon + \mathcal{O}(\varepsilon^2)$$

and

$$Re(\overline{p} \cdot B(\overline{q}, (2\lambda iI - A)^{-1}B(q, q))) = 0 + \mathcal{O}(\varepsilon^2).$$

Consequently we get

$$\ell_1(p_0) = -\frac{2}{45\sqrt{3}}\sqrt[n]{\left(\frac{n-2}{2}\right)^2}(n+7)(n-2) + \mathcal{O}(\varepsilon).$$

As we said before $\ell_1(p_0) < 0$ implies that we have a supercritical Hopf bifurcation at $\alpha = \alpha_{bif}$, so there exists $\varepsilon_0 > 0$ such that for $\alpha_{bif} < \alpha < \alpha_{bif} + \varepsilon_0$ the system (2) has a stable limit cycle.

3. Conclusions

The repressilator model is an implementation of an artificial oscillatory network used for studying the collections of interacting molecules in a cell. The model is given by a 6–dimensional differential system. O. Buse, A. Kuznetsov and R. Pérez published two nice papers (see [3, 4]) analysing a reduced system of dimension three. In this reduced system they show the existence of a supercritical Hopf bifurcation. Since the reduction is reasonable we may expect that such supercritical Hopf bifurcation must also occurs in the actual 6–dimensional differential system. We prove that this is the case.

4. Appendix

The matrix $(2\sqrt{3}b\varepsilon iI - A)^{-1}$ is given by

$$
(2\sqrt{3}b\varepsilon iI - A)^{-1} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix},
$$

where $C_{ij} = A_{ij} + B_{ij}$, for $i, j = 1, 2, \ldots, 6$.

$$
\begin{aligned}
A_{11} &= \frac{1}{189}\left(213 - 16i\sqrt{3}\right), \\
A_{12} &= \frac{2}{189}\left(-9 + 34i\sqrt{3}\right), \\
A_{13} &= \frac{4}{189}\left(-15 - 4i\sqrt{3}\right), \\
A_{14} &= \frac{4\left(12\left(\sqrt{3} - 124i\right)b\epsilon - \left(124\sqrt{3} + 3i\right)\sigma\epsilon + 72\sqrt{3} + 18i\right)}{81\left(10\sqrt{3} + 27i\right)b\epsilon}, \\
A_{15} &= \frac{3156\sqrt{3}b\epsilon + 6144ib\epsilon + 512\sqrt{3}\sigma\epsilon - 789i\sigma\epsilon - 864\sqrt{3} + 666i}{810\sqrt{3}b\epsilon + 2187ib\epsilon}, \\
A_{16} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon}, \\
A_{21} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon}, \\
A_{22} &= \frac{1}{189}\left(213 - 16i\sqrt{3}\right), \\
A_{23} &= \frac{1}{189}\left(213 - 16i\sqrt{3}\right), \\
A_{24} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon}, \\
A_{25} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon}, \\
A_{26} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon}, \\
A_{31} &= \frac{4\left(12\left(50 + 87i\sqrt{3}\right)b\epsilon + \left(261 - 50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27 - 10i\sqrt{3}\right)b\epsilon},
\end{aligned}
$$

$$A_{32} = \frac{4\left(12\left(50+87i\sqrt{3}\right)b\epsilon + \left(261-50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27-10i\sqrt{3}\right)b\epsilon},$$
$$A_{33} = \frac{4\left(12\left(50+87i\sqrt{3}\right)b\epsilon + \left(261-50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27-10i\sqrt{3}\right)b\epsilon},$$
$$A_{34} = \frac{4\left(12\left(50+87i\sqrt{3}\right)b\epsilon + \left(261-50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27-10i\sqrt{3}\right)b\epsilon},$$
$$A_{35} = \frac{4\left(12\left(50+87i\sqrt{3}\right)b\epsilon + \left(261-50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27-10i\sqrt{3}\right)b\epsilon},$$
$$A_{36} = \frac{4\left(12\left(50+87i\sqrt{3}\right)b\epsilon + \left(261-50i\sqrt{3}\right)\sigma\epsilon + 18i\sqrt{3} - 225\right)}{81\left(27-10i\sqrt{3}\right)b\epsilon},$$
$$A_{41} = \frac{1}{189}\left(9-34i\sqrt{3}\right),$$
$$A_{42} = \frac{2}{189}\left(15+4i\sqrt{3}\right),$$
$$A_{43} = \frac{4}{189}\left(-3+2i\sqrt{3}\right),$$
$$A_{44} = -\frac{912\sqrt{3}b\epsilon + 480ib\epsilon + 40\sqrt{3}\sigma\epsilon - 228i\sigma\epsilon - 432\sqrt{3} + 333i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$A_{45} = \frac{1188\sqrt{3}b\epsilon - 984ib\epsilon - 82\sqrt{3}\sigma\epsilon - 297i\sigma\epsilon + 36\sqrt{3} + 450i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$A_{46} = \frac{4\left(12\left(\sqrt{3}+44i\right)b\epsilon + \left(44\sqrt{3}-3i\right)\sigma\epsilon - 9\left(4\sqrt{3}+i\right)\right)}{81\left(10\sqrt{3}+27i\right)b\epsilon},$$
$$A_{51} = \frac{4}{189}\left(-3+2i\sqrt{3}\right),$$
$$A_{52} = \frac{4}{189}\left(-3+2i\sqrt{3}\right),$$
$$A_{53} = \frac{2}{189}\left(15+4i\sqrt{3}\right),$$
$$A_{54} = \frac{4\left(12\left(\sqrt{3}+44i\right)b\epsilon + \left(44\sqrt{3}-3i\right)\sigma\epsilon - 9\left(4\sqrt{3}+i\right)\right)}{81\left(10\sqrt{3}+27i\right)b\epsilon},$$
$$A_{55} = -\frac{912\sqrt{3}b\epsilon + 480ib\epsilon + 40\sqrt{3}\sigma\epsilon - 228i\sigma\epsilon - 432\sqrt{3} + 333i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$A_{56} = \frac{1188\sqrt{3}b\epsilon - 984ib\epsilon - 82\sqrt{3}\sigma\epsilon - 297i\sigma\epsilon + 36\sqrt{3} + 450i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$A_{61} = \frac{2}{189}\left(15+4i\sqrt{3}\right),$$
$$A_{62} = \frac{4}{189}\left(-3+2i\sqrt{3}\right),$$
$$A_{63} = \frac{1}{189}\left(9-34i\sqrt{3}\right),$$
$$A_{64} = \frac{1188\sqrt{3}b\epsilon - 984ib\epsilon - 82\sqrt{3}\sigma\epsilon - 297i\sigma\epsilon + 36\sqrt{3} + 450i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$

$$A_{65} = \frac{1188\sqrt{3}b\epsilon - 984ib\epsilon - 82\sqrt{3}\sigma\epsilon - 297i\sigma\epsilon + 36\sqrt{3} + 450i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$A_{66} = \frac{1188\sqrt{3}b\epsilon - 984ib\epsilon - 82\sqrt{3}\sigma\epsilon - 297i\sigma\epsilon + 36\sqrt{3} + 450i}{810\sqrt{3}b\epsilon + 2187ib\epsilon},$$
$$B_{11} = \frac{2\epsilon\left(\left(-4800 - 5921i\sqrt{3}\right)b + \left(-362 + 316i\sqrt{3}\right)\sigma\right)}{3969},$$
$$B_{12} = \frac{\epsilon\left(8\left(215\sqrt{3} + 42i\right)b - \left(2\sqrt{3} + 259i\right)\sigma\right)}{81\left(4\sqrt{3} + i\right)},$$
$$B_{13} = \frac{\epsilon\left(8\left(215\sqrt{3} + 42i\right)b - \left(2\sqrt{3} + 259i\right)\sigma\right)}{81\left(4\sqrt{3} + i\right)},$$
$$B_{14} = \frac{2\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(96\Big(444\sqrt{3} - 131i\Big)b^2 - 52\Big(41\sqrt{3} + 324i\Big)b\sigma$$
$$+ \left(-516\sqrt{3} + 271i\right)\sigma^2\right),$$
$$B_{15} = \frac{\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(24\Big(-1760\sqrt{3} + 5807i\Big)b^2 + 2\Big(9247\sqrt{3} + 5952i\Big)b\sigma$$
$$+8\left(14\sqrt{3} - 215i\right)\sigma^2\right),$$
$$B_{16} = \frac{\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(-48\Big(1058\sqrt{3} + 2825i\Big)b^2 + 16\Big(-1021\sqrt{3}$$
$$+1362i\big)b\sigma + \left(758\sqrt{3} + 1259i\right)\sigma^2\big),$$
$$B_{21} = \frac{4\epsilon\left(\left(-146\sqrt{3} + 528i\right)b + \left(32\sqrt{3} + 23i\right)\sigma\right)}{81\left(4\sqrt{3} + i\right)},$$
$$B_{22} = \frac{2\epsilon\left(\left(-4800 - 5921i\sqrt{3}\right)b + \left(-362 + 316i\sqrt{3}\right)\sigma\right)}{3969},$$
$$B_{23} = \frac{\epsilon\left(8\left(215\sqrt{3} + 42i\right)b - \left(2\sqrt{3} + 259i\right)\sigma\right)}{81\left(4\sqrt{3} + i\right)},$$
$$B_{24} = \frac{\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(-48\Big(1058\sqrt{3} + 2825i\Big)b^2 + 16\Big(-1021\sqrt{3}$$
$$+1362i\big)b\sigma + \left(758\sqrt{3} + 1259i\right)\sigma^2\big),$$
$$B_{25} = \frac{2\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(96\Big(444\sqrt{3} - 131i\Big)b^2$$
$$-52\left(41\sqrt{3} + 324i\right)b\sigma + \left(-516\sqrt{3} + 271i\right)\sigma^2\big),$$
$$B_{26} = \frac{\epsilon}{81\left(2\sqrt{3} - 3i\right)^3 b}\Big(24\Big(-1760\sqrt{3} + 5807i\Big)b^2 + 2\Big(9247\sqrt{3}$$
$$+5952i\big)b\sigma + 8\left(14\sqrt{3} - 215i\right)\sigma^2\big),$$
$$B_{31} = \frac{\epsilon\left(8\left(215\sqrt{3} + 42i\right)b - \left(2\sqrt{3} + 259i\right)\sigma\right)}{81\left(4\sqrt{3} + i\right)},$$
$$B_{32} = \frac{2\epsilon\left(\left(-4800 - 5921i\sqrt{3}\right)b + \left(-362 + 316i\sqrt{3}\right)\sigma\right)}{3969},$$
$$B_{33} = \frac{2\epsilon\left(\left(-4800 - 5921i\sqrt{3}\right)b + \left(-362 + 316i\sqrt{3}\right)\sigma\right)}{3969},$$

$$B_{34} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(24\left(-1760\sqrt{3}+5807i\right)b^2+2\Big(9247\sqrt{3}$$
$$+5952i\Big)b\sigma+8\left(14\sqrt{3}-215i\right)\sigma^2\Big),$$
$$B_{35} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(-48\left(1058\sqrt{3}+2825i\right)b^2+16\Big(-1021\sqrt{3}$$
$$+1362i\Big)b\sigma+\left(758\sqrt{3}+1259i\right)\sigma^2\Big),$$
$$B_{36} = \frac{2\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(96\left(444\sqrt{3}-131i\right)b^2-52\left(41\sqrt{3}+324i\right)b\sigma$$
$$+\left(-516\sqrt{3}+271i\right)\sigma^2\Big),$$
$$B_{41} = \frac{2\epsilon\left(-259\sqrt{3}b+6ib+8\sqrt{3}\sigma+22i\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{42} = -\frac{\epsilon\left(8\left(-23\sqrt{3}+96i\right)b+\left(40\sqrt{3}+19i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{43} = \frac{4\epsilon\left(2\left(35\sqrt{3}+54i\right)b+\left(6\sqrt{3}-13i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{44} = \frac{2\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(96\left(14\sqrt{3}-215i\right)b^2-52\left(53\sqrt{3}+6i\right)b\sigma$$
$$+\left(2\sqrt{3}+259i\right)\sigma^2\Big),$$
$$B_{45} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(24\left(758\sqrt{3}+1259i\right)b^2+2\Big(1627\sqrt{3}$$
$$-3810i\Big)b\sigma-8\left(32\sqrt{3}+23i\right)\sigma^2\Big),$$
$$B_{46} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(48\left(-516\sqrt{3}+271i\right)b^2+16\Big(131\sqrt{3}$$
$$+576i\Big)b\sigma+\left(252\sqrt{3}-253i\right)\sigma^2\Big),$$
$$B_{51} = \frac{4\epsilon\left(2\left(35\sqrt{3}+54i\right)b+\left(6\sqrt{3}-13i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{52} = \frac{2\epsilon\left(-259\sqrt{3}b+6ib+8\sqrt{3}\sigma+22i\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{53} = -\frac{\epsilon\left(8\left(-23\sqrt{3}+96i\right)b+\left(40\sqrt{3}+19i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},$$
$$B_{54} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(48\left(-516\sqrt{3}+271i\right)b^2+16\Big(131\sqrt{3}$$
$$+576i\Big)b\sigma+\left(252\sqrt{3}-253i\right)\sigma^2\Big),$$
$$B_{55} = \frac{2\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(96\left(14\sqrt{3}-215i\right)b^2-52\left(53\sqrt{3}+6i\right)b\sigma$$
$$+\left(2\sqrt{3}+259i\right)\sigma^2\Big),$$
$$B_{56} = \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(24\left(758\sqrt{3}+1259i\right)b^2+2\Big(1627\sqrt{3}$$
$$-3810i\Big)b\sigma-8\left(32\sqrt{3}+23i\right)\sigma^2\Big),$$

$$
\begin{aligned}
B_{61} &= -\frac{\epsilon\left(8\left(-23\sqrt{3}+96i\right)b+\left(40\sqrt{3}+19i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},\\
B_{62} &= \frac{4\epsilon\left(2\left(35\sqrt{3}+54i\right)b+\left(6\sqrt{3}-13i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},\\
B_{63} &= \frac{4\epsilon\left(2\left(35\sqrt{3}+54i\right)b+\left(6\sqrt{3}-13i\right)\sigma\right)}{81\left(4\sqrt{3}+i\right)},\\
B_{64} &= \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(24\Big(758\sqrt{3}+1259i\Big)b^2+2\Big(1627\sqrt{3}\\
&\qquad -3810i\big)b\sigma-8\left(32\sqrt{3}+23i\right)\sigma^2\Big),\\
B_{65} &= \frac{\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(48\Big(-516\sqrt{3}+271i\Big)b^2+16\Big(131\sqrt{3}\\
&\qquad +576i\big)b\sigma+\left(252\sqrt{3}-253i\right)\sigma^2\Big) \quad \text{and}\\
B_{66} &= \frac{2\epsilon}{81\left(2\sqrt{3}-3i\right)^3 b}\Big(96\Big(14\sqrt{3}-215i\Big)b^2-52\Big(53\sqrt{3}+6i\Big)b\sigma\\
&\qquad +\left(2\sqrt{3}+259i\right)\sigma^2\Big).
\end{aligned}
$$

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