No periodic orbits for the type A Bianchi's systems

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Abstract We prove that all the type A Bianchi's systems do not have periodic solutions.

Keywords Bianchi cosmological models · periodic orbits

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1 Introduction

This paper deals with the Bianchi's cosmological models. These models require a three dimensional Lie algebra and Bianchi [1,2] was the first to solve the problem of classifying three dimensional Lie algebras. There are nine types of models according with the dimension n of the algebra.

a) $n = 0$ Type I;
b) $n = 1$ Type II, III;
c) $n = 2$ Type IV, V, VI, VII;
d) $n = 3$ Type $VIII$, IX.

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Let $\{X_1,\ X_2,\ X_3\}$ be an appropriate basis of the three dimensional Lie Algebra. The classification depends on a scalar $a \in \mathbb{R}$ and a vector (n_1, n_2, n_3) with $n_i \in \{+1, -1, 0\}$ such that

$$[X_1, X_2] = n_3 X_3, \quad [X_2, X_3] = n_1 X_1 - a X_2, \quad [X_3, X_1] = n_2 X_2 + a X_1,$$

where $[,]$ is the Lie bracket. In particular, for $a = 0$ we obtain models of type A and for $a \neq 0$ we obtain models of type B. For more details see Bogoyavlensky [3].

Type	I	II	VI_0	VII_0	$VIII$	IX
n_1	0	1	1	1	1	1
n_2	0	0	-1	1	1	1
n_3	0	0	0	0	-1	1

Table 1 Cosmologies of types A

According with [3] all cases of type A are Hamiltonian systems in the phase space p_i, q_i for $i = 1, 2, 3$ with the Hamiltonian function

$$H = \frac{1}{(q_1 q_2 q_3)^{\frac{1-k}{2}}} \left(T + \frac{1}{4} V_G \right),$$

with $0 \leq k \leq 1$ and where

$$T = 2 \sum_{i<j}^{3} p_i p_j q_i q_j - \sum_{i=1}^{3} p_i^2 q_i^2, \text{ and } V_G = 2 \sum_{i<j}^{3} n_i n_j q_i q_j - \sum_{i=1}^{3} n_i^2 q_i^2,$$

for $i, j, k = 1, 2, 3$.

If we rescale the time by τ defined by $d\tau = (q_1q_2q_3)^{-k/2}dt$ then the system becomes

$$
\begin{aligned}
\dot{q}_1 &= \frac{2q_1}{(q_1q_2q_3)^{(1-k)/2}}(p_2q_2 + p_3q_3 - p_1q_1),\\
\dot{q}_2 &= \frac{2q_2}{(q_1q_2q_3)^{(1-k)/2}}(p_3q_3 + p_1q_1 - p_2q_2),\\
\dot{q}_3 &= \frac{2q_3}{(q_1q_2q_3)^{(1-k)/2}}(p_1q_1 + p_2q_2 - p_3q_3),\\
\dot{p}_1 &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\,(2p_1(p_2q_2 + p_3q_3 - p_1q_1)+\\
&\quad \frac{1}{2}n_1(n_2q_2 + n_3q_3 - n_1q_1)) + \frac{1-k}{2}\frac{\overline{H}}{q_1},\\
\dot{p}_2 &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\,(2p_2(p_3q_3 + p_1q_1 - p_2q_2)+\\
&\quad \frac{1}{2}n_2(n_3q_3 + n_1q_1 - n_2q_2)) + \frac{1-k}{2}\frac{\overline{H}}{q_2},\\
\dot{p}_3 &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\,(2p_3(p_1q_1 + p_2q_2 - p_3q_3)+\\
&\quad \frac{1}{2}n_3(n_1q_1 + n_2q_2 - n_3q_3)) + \frac{1-k}{2}\frac{\overline{H}}{q_3},
\end{aligned}
\tag{1}
$$

with $\overline{H} = T + V_G/4$. The constants n_1, n_2, n_3 determine the type of the model according with Table 1. Performing the change of coordinates $ds = (q_1q_2q_3)^{\frac{1-k}{2}}d\tau$ and $q_i = x_i$ and $p_i = x_{i+3}/x_i$, $i = 1, 2, 3$, we obtain the system

$$
\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= n_1x_1(n_1x_1 - n_2x_2 - n_3x_3) + \tfrac{1}{4}(k-1)F_2,\\
\dot{x}_5 &= n_2x_1(-n_1x_1 + n_2x_2 - n_3x_3) + \tfrac{1}{4}(k-1)F_2,\\
\dot{x}_6 &= n_3x_3(-n_1x_1 - n_2x_2 + n_3x_3) + \tfrac{1}{4}(k-1)F_2,
\end{aligned}
\tag{2}
$$

with

$$
\begin{aligned}
F_2 = {}& n_1^2x_1^2 + n_2^2x_2^2 + n_3^2x_3^2 - 2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 - 2n_2n_3x_2x_3\\
&+ x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6.
\end{aligned}
$$

Note that system (2) is a homogeneous system of degree 2, and the first integral given by the Hamiltonian H becomes

$$
\begin{aligned}
\mathcal{H} = {}& (x_1x_2x_3)^{\frac{k-1}{2}}(n_1^2x_1^2 + n_2^2x_2^2 + n_3^2x_3^2 - 2n_1n_2x_1x_2 - 2n_1n_3x_1x_3\\
&- 2n_2n_3x_2x_3 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6).
\end{aligned}
$$

It is known that all the Bianchi class A models do not have periodic orbits. This has been proved using evolutions equations associated to these models,

Fig. 1 Phase portrait of the differential equation (4).

and showing that such equations always have some monotone function evaluated on the orbits. Consequently these models cannot exhibit periodic motion. For more details, see chapter 6 of the book by Wainwright and Ellis [4].

In this article we provide a direct and easier proof on the non-existence of periodic orbit for the 6 models of Bianchi class A.

2 The Bianchi *I* system

In this section we consider the Bianchi I system. According with Table 1 we have $n_1 = n_2 = n_3 = 0$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \tfrac{k-1}{4}(x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6),\\
\dot{x}_5 &= \tfrac{k-1}{4}(x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6),\\
\dot{x}_6 &= \tfrac{k-1}{4}(x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6).
\end{aligned} \tag{3}$$

Proposition 1 *The Bianchi I system, given by* (3)*, does not have periodic solutions.*

Proof System (3) has the first integrals $F_1 = x_4 - x_5$ and $F_2 = x_4 - x_6$. Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of (3). So, there exist two constants a and b such that $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ for all t. We have that $x_4(t)$ is a periodic solution of the equation

$$\dot{x}_4 = \frac{k-1}{4}(-3x_4^2 - 2(a+b)x_4 + (a-b)^2). \tag{4}$$

Observe that the discriminant of the equation $-3x_4^2 - 2(a+b)x_4 + (a-b)^2 = 0$ is $\Delta = 4(a+b)^2 + 12(a-b)^2$. So $\Delta \geq 0$, if $\Delta > 0$ then equation (4) has two equilibrium points. One of them is an attractor and the other one is a repeller (see Figure $1(a)$). If $\Delta = 0$ then we have just one equilibrium point which is semi-stable (see Figure $1(b)$). In both cases the unique possibility in order to $x_4(t)$ be periodic is that $x_4(t) = c$ constant for all t being c an equilibrium of (4). Substituting $x_4(t) = c$, $x_5(t) = a + c$ and $x_6(t) = b + c$ in the first three equations of (3), and using that $x_1(t)$, $x_2(t)$ and $x_3(t)$ are periodic, we get that $x_1(t)$, $x_2(t)$ and $x_3(t)$ must be constant for all t. So $\Gamma(t)$ is constant, i.e. $\Gamma(t)$ is an equilibrium point of (3), and (3) do not have periodic solutions.

3 The Bianchi II system

In this section we consider the Bianchi II system. According with Table 1 we have $n_1 = 1$ and $n_2 = n_3 = 0$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= x_1^2 + \tfrac{k-1}{4}F,\\
\dot{x}_5 &= \tfrac{k-1}{4}F,\\
\dot{x}_6 &= \tfrac{k-1}{4}F,
\end{aligned} \tag{5}$$

where $F = x_1^2 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$.

Proposition 2 *The Bianchi II system, given by* (5)*, does not have periodic solutions.*

Proof Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of (5). The real function $x_4(t) - x_5(t)$ is periodic. So, it is bounded. If it is not constant then there exists t_0 such that $\dot{x}_4(t_0) - \dot{x_5}(t_0) < 0$, but from (5) we have that $\dot{x}_4(t_0) - \dot{x}_5(t_0) = (x_1(t_0))^2$. It implies that there exists a constant $a \in \mathbb{R}$ such that $x_5(t) = x_4(t) + a$ and $x_1(t) = 0$ for all t. By using the same argument we have that there exists another constant $b \in \mathbb{R}$ such that $x_6(t) = x_4(t) + b$ for all t. Next step is substitute $x_1(t) = 0$, $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ in the equation $\dot{x}_4 = x_1^2 + \frac{k-1}{4}F$, we get $\dot{x}_4 = -3x_4^2 - (a+b)x_4 + (a-b)^2$. By using the same argument of the proof of Proposition 1 we conclude the proof of this proposition.

4 The Bianchi VI_0 system

In this section we consider the Bianchi VI_0 system. According with Table 1 we have $n_1 = 1$, $n_2 = -1$ and $n_3 = 0$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= x_1(x_1 + x_2) + \tfrac{k-1}{4}F,\\
\dot{x}_5 &= x_2(x_1 + x_2) + \tfrac{k-1}{4}F,\\
\dot{x}_6 &= \tfrac{k-1}{4}F,
\end{aligned} \tag{6}$$

where $F = x_1^2 + x_2^2 + 2x_1x_2 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$.

Proposition 3 *The Bianchi VI_0 system, given by* (6)*, does not have periodic solutions.*

Proof Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of (6). The real function $x_4(t) + x_5(t) - 2x_6(t)$ is periodic. So, it is bounded. If it is not constant then there exists t_0 such that $\dot{x}_4(t_0) + \dot{x}_5(t_0) - 2\dot{x}_6(t_0) < 0$, but from (6) we have that $\dot{x}_4(t_0) + \dot{x}_5(t_0) - 2\dot{x}_6(t_0) = (x_1(t_0) + x_2(t_0))^2$. It implies that $\dot{x}_4(t) + \dot{x}_5(t) - 2\dot{x}_6(t) = 0 = (x_1(t) + x_2(t))^2$ for all t. Substituting $x_1(t) = -x_2(t)$ in (6) we have that $\dot{x}_4(t) - \dot{x}_5(t) = 0$ and $\dot{x}_4(t) - \dot{x}_6(t) = 0$ for all t. There exist constants $a, b \in \mathbb{R}$ such that $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ for all t. Next step is substitute $x_1(t) = -x_2(t)$, $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ in the equation $\dot{x}_4 = x_1(x_1 + x_2) + \frac{k-1}{4}F$. By using the same argument of the proof of Proposition 1 we conclude the proof of this proposition.

5 The Bianchi VII_0 system

In this section we consider the Bianchi VII_0 system. According with Table 1 we have $n_1 = 1$ and $n_2 = 1$ and $n_3 = 0$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= x_1(x_1 - x_2) + \tfrac{k-1}{4}F,\\
\dot{x}_5 &= x_2(-x_1 + x_2) + \tfrac{k-1}{4}F,\\
\dot{x}_6 &= \tfrac{k-1}{4}F,
\end{aligned} \tag{7}$$

where $F = x_1^2 + x_2^2 - 2x_1x_2 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$.

Proposition 4 *The Bianchi VII_0 system, given by* (7)*, does not have periodic solutions.*

Proof Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of (7). The real function $x_4(t) + x_5(t) - 2x_6(t)$ is periodic. So, it is bounded. If it is not constant then there exists t_0 such that $\dot{x}_4(t_0) + \dot{x}_5(t_0) - 2\dot{x}_6(t_0) < 0$, but from (7) we have that $\dot{x}_4(t_0) + \dot{x}_5(t_0) - 2\dot{x}_6(t_0) = (x_1(t_0) - x_2(t_0))^2$. It implies that $\dot{x}_4(t) + \dot{x}_5(t) - 2\dot{x}_6(t) = 0 = (x_1(t) - x_2(t))^2$ for all t. Substituting $x_1(t) = x_2(t)$ in (7) we have that $\dot{x}_4(t) - \dot{x}_5(t) = 0$ and $\dot{x}_4(t) - \dot{x}_6(t) = 0$ for all t. There exist constants $a, b \in \mathbb{R}$ such that $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ for all t. Next step is substitute $x_1(t) = x_2(t)$, $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ in the equation $\dot{x}_4 = x_1(x_1 - x_2) + \frac{k-1}{4}F$. By using the same argument of the proof of Proposition 1 we conclude the proof of this proposition.

6 The Bianchi $VIII$ system

In this section we consider the Bianchi $VIII$ system. According with Table 1 we have $n_1 = 1$ and $n_2 = 1$ and $n_3 = -1$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= x_1(x_1 - x_2 + x_3) + \tfrac{k-1}{4}F,\\
\dot{x}_5 &= x_2(-x_1 + x_2 + x_3) + \tfrac{k-1}{4}F,\\
\dot{x}_6 &= x_3(x_1 + x_2 + x_3) + \tfrac{k-1}{4}F,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
F = {}& x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3 +\\
& x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6.
\end{aligned} \tag{9}$$

Lemma 1 *The hyperplanes* $\{x_1 = 0\}$, $\{x_2 = 0\}$, *and* $\{x_3 = 0\}$ *are invariant manifolds for system* (8) *and there is no periodic orbits in these hyperplanes.*

Proof Clearly the hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$, and $\{x_3 = 0\}$ are invariant manifolds for system (8), i.e. if a solution of (8) has a point in $\{x_i = 0\}$ then the whole solution is contained in $\{x_i = 0\}$. Now we prove that in the hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$, and $\{x_3 = 0\}$ there are no periodic orbits. Let $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t),\ x_5(t), x_6(t))$ be a periodic solution of (8). Suppose that $\Gamma(t)$ is in $\{x_1 = 0\}$. From (8) we have $\dot{x}_5 + \dot{x}_6 - 2\dot{x}_4 = (x_2 + x_3)^2$. We get that $x_1(t) = 0$ and $x_2(t) = -x_3(t)$ for all t. Substituting these conditions in the equations of (8) we have that $\dot{x}_4 = \dot{x}_5 = \dot{x}_6$, and so there exist constants a and b such that $x_5(t) = x_4(t) + a$ and $x_6(t) = x_4(t) + b$ for all t. Substituting all these conditions in the fourth equation of (8) we obtain again the equation (4). So, in order that Γ be periodic, $x_4(t)$ is constant. Now from the second and third equations of (8) we have that $x_2(t)$ and $x_3(t)$ are constants, and Γ is an equilibrium point instead of a periodic orbit. In the same way we can prove that there are no periodic orbits in $\{x_2 = 0\}$ and in $\{x_3 = 0\}$.

Consider the three sets

$$\begin{aligned}
F^+ &= \{x \in \mathbb{R}^6 : F(x) > 0\},\\
F^0 &= \{x \in \mathbb{R}^6 : F(x) = 0\} \text{ and}\\
F^- &= \{x \in \mathbb{R}^6 : F(x) < 0\},
\end{aligned}$$

where F is given in (9).

Lemma 2 *The sets* F^+, F^0 *and* F^- *are invariant by system* (8) *and there are no periodic orbits in the set* F^-.

Proof First of all observe that if we call X the vector field associated to the system (8) then we have that

$$XF = \langle X, \nabla F \rangle = -\frac{1}{2}(k-1)(x_4 + x_5 + x_6)F, \tag{10}$$

where $\langle .,. \rangle$ denotes the standard inner product in $\mathbb{R}^6$ and ∇F is the gradient of F. From (10) we get that F^0 is an invariant set to (8), and consequently F^+ and F^- also are invariant.

Now we prove that there are not periodic orbit in F^-. Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic orbit of (8) and that it is in the set F^-. By Lemma 1 we have that $x_1(t) \neq 0$, $x_2(t) \neq 0$ and $x_3(t) \neq 0$ for all t. Consider the function

$$h(x) = \frac{x_4 + x_5 + x_6}{x_1 x_2 x_3}.$$

Observe that $h \circ \Gamma$ is defined for all t and it is a periodic function. So there exists at least a point $t = t_0$ such that $(h \dot{\circ} \Gamma)(t_0) = 0$. We have

$$(h \dot{\circ} \Gamma)(t) = \langle \nabla h, \dot{\Gamma}(t) \rangle = Xh(\Gamma(t)).$$

By the other hand

$$Xh(x) = (1 + 3k)F(x) - 8(x_4^2 + x_5^2 + x_6^2), \tag{11}$$

which is always negative in the set F^-. So the periodic orbit $\Gamma(t)$ cannot be contained in F^-.

Let U be a subset of $\mathbb{R}^6$. Let $h : U \to \mathbb{R}$ be a C^1 function. By $S(h)$ we denote the set $\{x \in \mathbb{R}^6 : Xh(x) = 0\}$. Suppose that we are interested in the localization of the periodic orbits of system $\dot{x} = X(x)$ located in the set U. We define $h_{inf} = inf\{h(x) : x \in U \cap S(h)\}$, $h_{sup} = sup\{h(x) : x \in U \cap S(h)\}$. The following two propositions are particular cases of results that can be found in [5].

Proposition 5 *All the periodic orbits of system $\dot{x} = X(x)$ located in U are contained in the set $\{x \in U : h_{inf} \leq h(x) \leq h_{sup}\}$.*

Proof Let $\Gamma(t)$ be a periodic orbit of system $\dot{x} = X(x)$ contained in the set U. Denote by $\gamma = \{\Gamma(t) : t \in \mathbb{R}\}$. The set γ is compact and so the C^1 function h, restricted to the set γ, has a maximum M and a minimum m. In particular $\gamma \subset \{x \in U : m \leq h(x) \leq M\}$. For all points $t = t_1$ such that $h(\Gamma(t_1)) = m$ we have that $\Gamma(t_1) \in S(h)$. It implies that $m = inf\{h(x) : x \in \gamma \cap S(h)\}$. On the other hand we have that $\gamma \cap S(h) \subset U \cap S(h)$ implies $m \geq h_{inf}$. Analogously we have $h_{sup} \geq M$. So we have

$$\gamma \subset \{x \in U : m \leq h(x) \leq M\} \subset \{x \in U : h_{inf} \leq h(x) \leq h_{sup}\}.$$

Proposition 6 *Let U be a set in $\mathbb{R}^6$. If $S(h) \cap U = \emptyset$ then system $\dot{x} = X(x)$ has no periodic orbits contained in U.*

Proof Suppose that $\Gamma(t)$ is a periodic orbit of system $\dot{x} = X(x)$ contained in the set U. As we saw in the proof of Proposition 5, for all points $t = t_1$ such that $h(\Gamma(t_1)) = m$ we have that $\Gamma(t_1) \in S(h) \cap U$. And so $S(h) \cap U \neq \emptyset$, which is a contradiction.

Lemma 3 *There are no periodic orbits of system* (8) *located in* F^0.

Proof Consider the set $U = F^0 \cap \{x_1 \neq 0\} \cap \{x_2 \neq 0\} \cap \{x_3 \neq 0\}$ and the function

$$h(x) = \frac{x_4 + x_5 + x_6}{x_1 x_2 x_3}.$$

Accordingly to (11) we have that $S(h) = \{x \in \mathbb{R}^6 : x_4 = x_5 = x_6 = 0\}$. So $h_{inf} = h_{sup} = 0$. From Proposition 5 all compact invariant sets of (8) located on U are contained in $B = \{x_4 + x_5 + x_6 = 0\}$.

Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic orbit of (8) and it is in the set F^0. By using the fact that this orbit is contained in B we have that $\dot{x}_4 + \dot{x}_5 + \dot{x}_6 = 0$. From system (8) and the fact that Γ are in F^0 we have that $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 0$, and consequently $x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_4x_6 - 2x_5x_6 = 0$. Substituting $x_6 = -x_4 - x_5$ in the last equation we get

$$2x_4^2 + 2x_5^2 + 2(x_4 + x_5)^2 = 0,$$

and so $x_4(t) = x_5(t) = x_6(t) = 0$ for all t. Now substituting these values in system (8) we get that $x_1(t)$, $x_2(t)$ and $x_3(t)$ are constant functions. So Γ is not a periodic orbit.

Lemma 4 *If there exists a periodic orbit for system* (8)*, then it intersects the set* $\{x \in \mathbb{R}^6 : x_4 + x_5 + x_6 = 0\}$.

Proof Consider the set $B = \{x \in \mathbb{R}^6 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$ and $x_4 + x_5 + x_6 \neq 0\}$ and the function $h(x) = x_1x_2x_3$. We have that

$$Xh(x) = x_1x_2x_3(x_4 + x_5 + x_6),$$

and so $S(h) \cap B = \emptyset$. According with Proposition 6 system (8) has no periodic orbits in B. If system (8) has a periodic orbit then it intersects $\{x \in \mathbb{R}^6 : x_4 + x_5 + x_6 = 0\}$.

Lemma 5 *There are no periodic orbits for system* (8) *located in* F^+.

Proof Suppose that $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic orbit of system (8) and that it is in the set F^+. By Lemma 4 we have that Γ intersects the set $\{x_4 + x_5 + x_6 = 0\}$. Consider the function $h(x) = x_1x_2x_3(x_4 + x_5 + x_6)$. By Lemma 1 we have that $x_1(t) \neq 0$, $x_2(t) \neq 0$, and $x_3(t) \neq 0$ for all t. Observe that the zeroes of the function $h \circ \Gamma$ occur for the values t such that the orbit Γ intersects the set $\{x_4 + x_5 + x_6 = 0\}$. Computing the derivative of $h \circ \Gamma$ we have $(h \dot{\circ} \Gamma)(t) = \langle \nabla h, \dot{\Gamma}(t) \rangle = Xh(\Gamma(t))$ where

$$Xh(x) = (1 + 3k)F(x) + 16(x_4x_5 + x_4x_6 + x_5x_6).$$

We observe that in the set $\{x_4 + x_5 + x_6 = 0\}$ we have $x_4x_5 + x_4x_6 + x_5x_6 = 0$. We get that in all zeroes of the real periodic function $h \circ \Gamma$ its derivative is positive. This is a contradiction, because we cannot have a periodic real function with positive derivative in all of its zeroes.

Proposition 7 *The Bianchi $VIII$ system, given by* (8)*, does not have periodic solutions.*

Proof It follows from lemmas 2, 3, and 5.

7 The Bianchi IX system

In this section we consider the Bianchi IX system. According with Table 1 we have $n_1 = 1$ and $n_2 = 1$ and $n_3 = 1$. So system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= x_1(x_1 - x_2 - x_3) + \tfrac{k-1}{4}F,\\
\dot{x}_5 &= x_2(-x_1 + x_2 - x_3) + \tfrac{k-1}{4}F,\\
\dot{x}_6 &= x_3(x_1 + x_2 - x_3) + \tfrac{k-1}{4}F,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
F = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 +\\
x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6.
\end{aligned} \tag{13}$$

Lemma 6 *The hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$, and $\{x_3 = 0\}$ are invariant manifolds for system* (12) *and there are no periodic orbits in these hyperplanes.*

Proof The proof is very similar to the proof of Lemma 1.

Consider the three sets

$$\begin{aligned}
F^+ &= \{x \in \mathbb{R}^6 : F(x) > 0\},\\
F^0 &= \{x \in \mathbb{R}^6 : F(x) = 0\} \text{ and}\\
F^- &= \{x \in \mathbb{R}^6 : F(x) < 0\},
\end{aligned}$$

where F is given in (13).

Lemma 7 *The sets F^+, F^0 and F^- are invariant by system* (12) *and there are no periodic orbit in the set F^-.*

Proof The proof is very similar to the proof of Lemma 2.

Lemma 8 *There are no periodic orbits for system* (12) *located in F^0.*

Proof The proof is very similar to the proof of Lemma 3.

Lemma 9 *If there exists a periodic orbit for system* (12) *then it intersects the set $\{x \in \mathbb{R}^6 : x_4 + x_5 + x_6 = 0\}$.*

Proof It is the same proof of Lemma 4

Lemma 10 *There are no periodic orbits for system* (12) *located in F^+.*

Proof The same construction in the proof of Lemma 5 works in this case.

Proposition 8 *The Bianchi IX system, given by* (12)*, does not have periodic solutions.*

Proof It follows from lemmas 7, 8, and 10.

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