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Hopf bifurcation in the full repressilator equations

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In this paper, we prove that the full repressilator equations in dimension six undergo a supercritical Hopf bifurcation. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

Oscillatory networks are a particular kind of regulatory molecular networks, that is, collections of interacting molecules in a cell. The regulatory oscillators can be used to study abnormalities of a process in the cell, from sleep disorders to cancer. So, they attract significant attention among biologists and biophysicists. There are many implementations of artificial oscillatory networks (see, e.g., [1–7]). One of them is the repressilator [8]. Its genetic implementation uses three proteins that cyclically repress the synthesis of one another. The following system of DEs describes the behavior of the repressilator:

$$\begin{aligned} \dot m_1 &= -m_1 + \frac{\alpha}{1+v^n} + \alpha_0,\\ \dot m_2 &= -m_2 + \frac{\alpha}{1+w^n} + \alpha_0,\\ \dot m_3 &= -m_3 + \frac{\alpha}{1+u^n} + \alpha_0,\\ \dot u &= -\beta(u-m_1),\\ \dot v &= -\beta(v-m_2),\\ \dot w &= -\beta(w-m_3) \end{aligned} \tag{1}$$

Here, u, v, and w are proportional to the protein concentration, while m_i are proportional to the concentration of mRNA corresponding to those proteins. The nonlinear function $f(x) = \frac{\alpha}{1+x^n}$ reflects synthesis of the mRNAs from the DNA controlled by regulatory elements. The parameter α_0 represents uncontrolled part of the mRNA synthesis, and it is usually small. The explicit inclusion of the mRNA concentration variables into the model is given by β. Given that in general $\beta \ll 1$ and α_0 is very small, we consider $\alpha_0 = \varepsilon a$ and $\beta = \varepsilon b$, where a and b are positive constants and $\varepsilon > 0$ is sufficiently small. So, system (1) becomes

$$\begin{aligned} \dot m_1 &= -m_1 + \frac{\alpha}{1+v^n} + \varepsilon a,\\ \dot m_2 &= -m_2 + \frac{\alpha}{1+w^n} + \varepsilon a,\\ \dot m_3 &= -m_3 + \frac{\alpha}{1+u^n} + \varepsilon a,\\ \dot u &= -\varepsilon b(u-m_1),\\ \dot v &= -\varepsilon b(v-m_2),\\ \dot w &= -\varepsilon b(w-m_3) \end{aligned} \tag{2}$$

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