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RESEARCH ARTICLE

Bifurcation of limit cycles from a center in $\mathbb{R}^4$ in ressonance 1:N

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For every positive integer $N \geq 2$ we consider the linear differential center $\dot{x} = Ax$ in $\mathbb{R}^4$ with eigenvalues $\pm i$ and $\pm Ni$. We perturb this linear center inside the class of all polynomial differential systems of the form linear plus a homogeneous nonlinearity of degree N, i.e. $\dot{x} = Ax + \varepsilon F(x)$ where every component of $F(x)$ is a linear polynomial plus a homogeneous polynomial of degree N. Then if the displacement function of order ε of the perturbed system is not identically zero, we study the maximal number of limit cycles that can bifurcate from the periodic orbits of the linear differential center.

Keywords: periodic orbits, limit cycles, polynomial vector fields, perturbation, ressonance 1:N

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1. Introduction and statement of the main results

In the qualitative theory of differential equations the study of their limit cycles became one of the main topics. For a given differential equation $\mathcal{E}$ a *limit cycle* is a periodic orbit of $\mathcal{E}$ isolated in the set of all periodic orbits of $\mathcal{E}$.

Many questions arise on the limit cycles of the planar differential equations. Two main lines of research for such equations are, first the 16th Hilbert problem see for instance [3, 4], and second the study of how many limit cycles emerge from the periodic orbits of a center when we perturb it inside a given class of differential equations, see for example the book [2] and the references there in. More precisely the problem of consider the planar linear differential center

$$\dot{x} = -y, \qquad \dot{y} = x$$

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and perturb it

$$\dot{x} = -y + \varepsilon P(x,y), \qquad \dot{y} = x + \varepsilon Q(x,y),$$

inside a given class of polynomial differential equations and study the limit cycles bifurcating from the periodic orbits of the linear center has been attracted the interest and the research of many mathematicians. Of course ε is a small parameter. Here our main concerning is to bring this problem to higher dimension.

In this paper we consider the 4–dimensional linear differential center

$$\dot{x} = Ax, \tag{1}$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix},$$

with N a positive integer, and we perturb it

$$\dot{x} = Ax + \varepsilon F(x), \tag{2}$$

where ε is a small parameter and $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a polynomial of the form $F(x) = F_1(x) + F_N(x)$ with F_k a homogeneous polynomial of degree k in the variables $x = (x_1, x_2, x_3, x_4)$.

Our two main results are given in the next two theorems.

Theorem 1.1 : *Assume that $N \geq 2$ is even.*

(a) If $\varepsilon \neq 0$ is sufficiently small, then the maximum number of limit cycles of the differential equation (2) bifurcating from the periodic orbits of the linear differential center (1) if the displacement function of order ε is not identically zero is at most $2N$.
(b) For $\varepsilon \neq 0$ sufficiently small the differential equation

$$\begin{aligned} \dot{x}_1 &= \quad -x_2 + \varepsilon\left(2a_1x_1 - 2^N b_1 x_1^{N-2} x_2 x_3\right), \\ \dot{x}_2 &= \quad\;\; x_1 + \varepsilon\left(-2a_3x_1 - 2^N c_1 x_1^{N-2} x_2 x_3\right), \\ \dot{x}_3 &= -Nx_4 + \varepsilon\left(2a_2x_1 - 2^N b_2 x_1^{N-1} x_2\right), \\ \dot{x}_4 &= \quad Nx_3 + \varepsilon\, 2^N c_2 x_1^{N-1} x_2. \end{aligned} \tag{3}$$

has $2N$ limit cycles bifurcating from the periodic orbits of the linear differential center (1).

Theorem 1.1 is proved in Section 5. See Section 2 for the definition of displacement function of order ε.

Theorem 1.2 : *Assume that $N \geq 3$ is odd.*

(a) If $\varepsilon \neq 0$ is sufficiently small, then the maximum number of limit cycles of the differential equation (2) bifurcating from the periodic orbits of the linear differential center (1) if the displacement function of order ε is not identically zero is at most $N(N+2)$.

(b) For $\varepsilon \neq 0$ sufficiently small there are differential equations (2) *having* $N(N+2)$ *limit cycles bifurcating from the periodic orbits of the linear differential center* (1) *if* $N = 3, 5, 7, 9$.

Theorem 1.2 is proved in Section 6. Unfortunately we are not able to extend the results of statement (b) of Theorem 1.2 to all $N \geq 3$ odd, but we conjecture:

Conjecture. *Statement (b) of Theorem 1.2 holds for every $N \geq 3$ odd.*

We must remark that Theorem 1.2 has no meaning for $N = 1$ because then we are perturbing the linear differential center (1) inside the class of linear differential equations, and it is well known that linear differential equations have no limit cycles.

We note that the polynomial perturbation $F(x)$ of the form linear terms plus homogeneous nonlinearities of degree N that we are considering covers completely all the polynomial perturbations of system (2) of degree $N = 2, 3$. For $N = 2$ this is obvious, and for $N = 3$ this follows from the fact that using normal forms we can eliminate all the terms of degree 2 in the perturbation. For more details about these normal forms see [6].

In Section 2 we introduce the basic results on the averaging theory that we need for proving Theorems 1.1 and 1.2. The differential equation (2) is written into the normal form for applying the averaging theory in Section 3. In Section 4 we did the main computations related with the application of the averaging theory to our differential equations (2).

2. First order averaging theory

The aim of this section is to present the first order averaging method as it was obtained in [1]. Averaged function are given in terms of the Brouwer degree. In fact the Brouwer degree theory is the key point in the proof of this theorem. We remind here that continuity of some finite dimensional function is a sufficient condition for the existence of its Brouwer degree (see [5] for precise definitions).

Theorem 2.1: *We consider the following differential system*

$$\dot{x}(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{4}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$, are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $h : D \to \mathbb{R}^n$ as

$$h(z) = \int_0^T H(s, z) ds, \tag{5}$$

and assume that:

(i) H and R are locally Lipschitz with respect to x;

(ii) for $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in V \setminus \{a\}$ and $d_B(h, V, a) \neq 0$ (here $d_B(h, V, a)$ denote the Brouwer degree of h at a).

Then for $|\varepsilon| > 0$ sufficiently small there exists an isolated T–periodic solution $\varphi(\cdot, \varepsilon)$ of system (4) *such that $\varphi(a, 0) = a$.*

Here we will need some facts from the proof of Theorem 2.1. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0,T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot,z,\varepsilon)$ the solution of (4) with the initial value $x(0,z,\varepsilon)=z$. We consider also the function $\zeta : D\times(-\varepsilon_f,\varepsilon_f)\to\mathbb{R}^n$ defined by

$$\zeta(z,\varepsilon)=\int_0^T\Big(\varepsilon H\big(t,x(t,z,\varepsilon)\big)+\varepsilon^2R\big(t,x(t,z,\varepsilon),\varepsilon\big)\Big)dt.$$

From the proof of Theorem 2.1 we extract the following facts.

Remark 1 : Under the assumptions of Theorem 2.1 for every $z\in D$ the following relation holds

$$x(T,z,\varepsilon)-x(0,z,\varepsilon)=\zeta(z,\varepsilon).$$

Moreover the function ζ can be written in the form

$$\zeta(z,\varepsilon)=\varepsilon h(z)+\ \mathrm{O}\big(\varepsilon^2\big),$$

where h is given by (5) and the symbol $\mathrm{O}(\varepsilon^2)$ denotes a bounded function on every compact subset of $D\times(-\varepsilon_f,\varepsilon_f)$ multiplied by ε^2.

Note that from Remark **1** it follows that a zero z of the *displacement function* $\zeta(z,\varepsilon)$ at time T provides initial conditions for a periodic orbit of the system of period T. We also remark that $h(z)$ is the displacement function up to terms of order ε. Consequently the zeros of $h(z)$, when $h(z)$ is not identically zero, also provides periodic orbits of period T.

For a given system there is the possibility that the function ζ is not globally differentiable, but the function h is. In fact only differentiability in some neighborhood of a fixed isolated zero of f could be enough. When this is the case, one can use the following remark in order to verify the hypothesis (ii) of Theorem 2.1.

Remark 2 : Let $h : D\to\mathbb{R}^n$ be a C^1 function, with $h(a)=0$, where D is an open subset of $\mathbb{R}^n$ and $a\in D$. Whenever a is a simple zero of h (i.e. the Jacobian of f at a is not zero), then there exists a neighborhood V of a such that $h(z)\neq 0$ for all $z\in\overline{V}\setminus\{a\}$. Then $d_B(f,V,a)\in\{-1,1\}$.

3. Averaged System

Writing $F_1=\big(F_1^1,F_1^2,F_1^3,F_1^4\big)$ and $F_N=\big(F_N^1,F_N^2,F_N^3,F_N^4\big)$, system (2) becomes

$$\begin{aligned} x_1' &= -x_2+\varepsilon\big(F_1^1(x)+F_N^1(x)\big),\\ x_2' &= x_1+\varepsilon\big(F_1^2(x)+F_N^2(x)\big),\\ x_3' &= -Nx_4+\varepsilon\big(F_1^3(x)+F_N^3(x)\big),\\ x_4' &= Nx_3+\varepsilon\big(F_1^4(x)+F_N^4(x)\big). \end{aligned} \tag{6}$$

Lemma 3.1: *Changing the variables* (x_1,x_2,x_3,x_4) *to* (θ,r,ρ,s) *by*

$$\begin{aligned} &x_1=r\cos\theta, && x_2=r\sin\theta,\\ &x_3=\rho\cos\big(N(\theta+s)\big), && x_4=\rho\sin\big(N(\theta+s)\big), \end{aligned}$$

system (6) *is transformed into the system*

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \rho, s) + O(\varepsilon^2), \\ \frac{d\rho}{d\theta} &= \varepsilon H_2(\theta, r, \rho, s) + O(\varepsilon^2), \\ \frac{ds}{d\theta} &= \varepsilon H_3(\theta, r, \rho, s) + O(\varepsilon^2), \end{aligned} \tag{7}$$

where

$$\begin{aligned} H_1 &= \left(F_1^1 + F_N^1\right)\cos\theta + \left(F_1^2 + F_N^2\right)\sin\theta, \\ H_2 &= \left(F_1^3 + F_N^3\right)\cos\left(N(\theta+s)\right) + \left(F_1^4 + F_N^4\right)\sin\left(N(\theta+s)\right), \\ H_3 &= \tfrac{1}{N\rho}\Big(\left(F_1^4 + F_N^4\right)\cos\left(N(\theta+s)\right) - \left(F_1^3 + F_N^3\right)\sin\left(N(\theta+s)\right)\Big) - \\ &\quad \tfrac{1}{r}\Big(\left(F_1^2 + F_N^2\right)\cos\theta - \left(F_1^1 + F_N^1\right)\sin\theta\Big). \end{aligned}$$

Proof: System (6) in the variables (θ, r, ρ, s) becomes

$$\begin{aligned} \theta' &= 1 + \varepsilon\frac{1}{r}\Big(\cos\theta\left(F_1^2 + F_N^2\right) - \sin\theta\left(F_1^1 + F_N^1\right)\Big) \\ r' &= \varepsilon H_1(\theta, r, \rho, s), \\ \rho' &= \varepsilon H_2(\theta, r, \rho, s), \\ s' &= \varepsilon H_3(\theta, r, \rho, s). \end{aligned} \tag{8}$$

We notice that for $|\varepsilon|$ sufficiently small, $\theta'(t) > 0$ for each $(t, (\theta, r, \rho, s)) \in \mathbb{R} \times D$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is easy to see that the right hand side of the new system is well defined and continuous in $\mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f)$, it is 2π-periodic with respect to the independent variable θ and locally Lipschitz with respect to (r, ρ, s). From (8) is obtained after an expansion with respect to the small parameter ε. □

In what follows we assume that

$$F_1^g + F_N^g = \sum_{i+j+k+l=1} a_{ijkl}^g x_1^i x_2^j x_3^k x_4^l + \sum_{i+j+k+l=N} a_{ijkl}^g x_1^i x_2^j x_3^k x_4^l$$

for $g = 1, 2, 3, 4$.

Now we will prove a technical result that will need later on.

Lemma 3.2: *Let n be a non–negative integer and α and β be real numbers. The following statements hold.*

(a) $\cos^n\alpha = \displaystyle\sum_{i=0}^{[n/2]} b_i \cos\left((n-2i)\alpha\right)$;

(b) $\sin^n\alpha = \begin{cases} \displaystyle\sum_{i=0}^{n/2} b_i\cos\left((n-2i)\alpha\right) & \textit{if } n \textit{ is even}, \\ \displaystyle\sum_{i=0}^{(n-1)/2} b_i\sin\left((n-2i)\alpha\right) & \textit{if } n \textit{ is odd}. \end{cases}$

(c) The expression $\cos^i\alpha\sin^j\alpha\cos^k\beta\sin^l\beta$, *where* i, j, k *and* l *are non–negative*

integers, is equal to

$$\sum_{m=0}^{[\frac{i+j}{2}]}\sum_{M=0}^{[\frac{k+l}{2}]} c_{mM}\cos\Big(\big((i+j-2m)\alpha\big)\pm\big((k+l-2M)\beta\big)\Big),$$

or

$$\sum_{m=0}^{[\frac{i+j}{2}]}\sum_{M=0}^{[\frac{k+l}{2}]} d_{mM}\sin\Big(\big((i+j-2m)\alpha\big)\pm\big((k+l-2M)\beta\big)\Big),$$

if $j+l$ is even or odd, respectively.

Proof: The proof of the statements (a) and (b) follows immediately from the identities:

(I_1) $2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$,
(I_2) $2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$,
(I_3) $2\cos\alpha\sin\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$.

In order to analyze the expression given in (c), from statements (a) and (b), we have to consider only two cases: (1) j and l odd; (2) $j+l$ odd. This is because in the others cases it is a sum of cosines.

(1) We consider $j=2\mu+1$ and $l=2\sigma+1$ and rewrite the product as

$$\cos^i\alpha\sin^{2\mu}\alpha\cos^k\beta\sin^{2\sigma}\beta\sin\alpha\sin\beta.$$

The four first terms of this product are sum of cosines using again statements (a) and (b). Now using (I_2) we conclude the proof in this case.

(2) It is sufficient consider the case $j=2\mu+1$ and $l=2\delta$. So we have

$$\cos^i\alpha\sin^{2\mu}\alpha\cos^k\beta\sin^{2\sigma}\beta\sin\alpha.$$

By statements (a) and (b) the four first terms of this product are sum of cosines, and using (I_3) we conclude the proof. □

4. The functions $h_i(r,\rho,s)$ for $i=1,2,3$.

Now we shall compute the function $h(r,\rho,s) = \big(h_1(r,\rho,s),h_2(r,\rho,s),h_3(r,\rho,s)\big)$ given in (5) for our system (7).

Lemma 4.1: *The following statements hold.*

(a) If N is even then

$$h_1(r,\rho,s) = a_1 r + r^{N-1}\rho(b_1\sin Ns + c_1\cos Ns).$$

(b) If N is odd then

$$h_1(r,\rho,s) = a_1 r + r^{N-1}\rho(b_1\sin Ns + c_1\cos Ns) + \sum_{M=0}^{\frac{N-1}{2}} d^1_M r^{N-2M}\rho^{2M},$$

where a_1, b_1, c_1 and d^1_M's depend on the coefficients of the perturbation.

Proof: In order to organize the computations we write the function H_1 as

$$H_1 = H_1^1 + H_1^N = \big(F_1^1 \cos\theta + F_1^2 \sin\theta\big) + \big(F_N^1 \cos\theta + F_N^2 \sin\theta\big).$$

We will apply Theorem 2.1 to system (7). Next step is to find the function (5). Let h_1^1 be

$$\begin{aligned} h_1^1(r,\rho,s) &= \frac{1}{2\pi}\int_0^{2\pi} H_1^1(\theta,r,\rho,s)d\theta \\ &= \sum_{i+j+k+l=1} \frac{1}{2\pi}\int_0^{2\pi} \big(a_{ijkl}^1 x_1^i x_2^j x_3^k x_4^l \cos\theta + a_{ijkl}^2 x_1^i x_2^j x_3^k x_4^l \sin\theta\big)d\theta \\ &= \frac{a_{1000}^1 + a_{0100}^2}{2} r. \end{aligned}$$

Now we calculate

$$h_1^N(r,\rho,s) = \frac{1}{2\pi}\int_0^{2\pi} H_1^N(\theta,r,\rho,s)d\theta =$$

$$\sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^1 x_1^i x_2^j x_3^k x_4^l \cos\theta d\theta + \sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^2 x_1^i x_2^j x_3^k x_4^l \sin\theta d\theta =$$

$$\sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^1 r^{i+j}\rho^{k+l} \cos^{i+1}\theta \sin^j\theta \cos^k(N(\theta+s)) \sin^l(N(\theta+s))d\theta +$$

$$\sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^2 r^{i+j}\rho^{k+l} \cos^{i}\theta \sin^{j+1}\theta \cos^k(N(\theta+s)) \sin^l(N(\theta+s))d\theta.$$

By applying Lemma 3.2 we have that

$$h_1^N(r,\rho,s) = \sum_{i+j+k+l=N} r^{i+j}\rho^{k+l} \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=0}^{[\frac{i+j+1}{2}]} \sum_{M=0}^{[\frac{k+l}{2}]} C_{mM}^{ijkl}(\theta)d\theta,$$

where $C_{mM}^{ijkl}(\theta)$ is

$$\begin{aligned} &c_{mM}^{ijkl} \cos\Big(\big((i+j+1-2m)\theta\big) \pm \big((k+l-2M)N(\theta+s)\big)\Big) + \\ &d_{mM}^{ijkl} \sin\Big(\big((i+j+1-2m)\theta\big) \pm \big((k+l-2M)N(\theta+s)\big)\Big). \end{aligned}$$

All these integrals with respect to θ are zero except when

$$i+j+1-2m = N(k+l-2M). \tag{9}$$

Observe that $0 \le i+j+1-2m \le N+1$. So there are two possibilities: $k+l-2M = 0$ or $k+l-2M = 1$.

The first case to be considered is when N is even. Then we will show that the possibility $k+l-2M=0$ never occurs. If $k+l-2M=0$ then $k+l$ is even. So $i+j$ is even because N is even. It is a contradiction with (9). In short we have $k+l-2M=1$ and it implies from (9) that $N-(k+l)+1-2m=N$. So $k+l+2m=1$. Therefore $m=0$ and $k+l=1$, consequently $i+j=N-1$ and $M=0$. Finally we get $h_1^N(r,\rho,s)=r^{N-1}\rho(b_1\sin Ns+c_1\cos Ns)$ and this shows statement (a).

Now we consider the case N odd. When $k+l-2M=1$ we obtain the same that obtained in case N even, i.e. $r^{N-1}\rho(b_1\sin Ns+c_1\cos Ns)$. Now we consider the case $k+l-2M=0$. For each M, from 0 to $(N-1)/2$, we obtain the terms $d_M^1 r^{N-2M}\rho^{2M}$. So in this case we get

$$h_1^N(r,\rho,s)=r^{N-1}\rho(b_1\sin Ns+c_1\cos Ns)+\sum_{M=0}^{\frac{N-1}{2}} d_M^1 r^{N-2M}\rho^{2M},$$

and this proves statement (b). □

Lemma 4.2: *The following statements hold.*

(a) If N is even then

$$h_2(r,\rho,s)=a_2\rho+r^N(b_2\sin Ns+c_2\cos Ns).$$

(b) If N is odd then

$$h_2(r,\rho,s)=a_2\rho+r^N(b_2\sin Ns+c_2\cos Ns)+\sum_{M=0}^{\frac{N-1}{2}} d_M^2 r^{N-2M-1}\rho^{2M+1},$$

where a_2, b_2, c_2 and d_M^2's depend on the coefficients of the perturbation.

Proof: As in Lemma 4.1 we write the function H_2 as

$$H_2=H_2^1+H_2^N=\left(F_1^3+F_N^3\right)\cos\left(N(\theta+s)\right)+\left(F_1^4+F_N^4\right)\sin\left(N(\theta+s)\right).$$

Applying Theorem 2.1 to system (7) and using the same notation of Lemma 4.1 we get

$$\begin{aligned} h_2^1(r,\rho,s) &= \frac{1}{2\pi}\int_0^{2\pi} H_2^1(\theta,r,\rho,s)d\theta \\ &= \sum_{i+j+k+l=1}\frac{1}{2\pi}\int_0^{2\pi}\Big(a_{ijkl}^3 x_1^i x_2^j x_3^k x_4^l\cos\left(N(\theta+s)\right)+ \\ &\qquad\qquad a_{ijkl}^4 x_1^i x_2^j x_3^k x_4^l\sin\left(N(\theta+s)\right)\Big)d\theta \\ &= \frac{a_{0010}^3+a_{0001}^4}{2}\rho. \end{aligned}$$

Now we calculate $h_2^N(r,\rho,s)=\frac{1}{2\pi}\int_0^{2\pi}H_2^N(\theta,r,\rho,s)d\theta$ and obtain an expression similar to the one obtained in Lemma 4.1 except that the terms which the integrals are non necessarily zero are given by

$$i+j-2m=N(k+l+1-2M). \tag{10}$$

Observe that $0 \leq i+j-2m \leq N$. So there are two possibilities: $k+l+1-2M=0$ or $k+l+1-2M=1$.

Suppose N is even. Then we show that $k+l+1-2M=0$ never can occur. If it occurs then $k+l$ is odd. So $i+j$ is odd because N is even. It is a contradiction with (10). We have $k+l+1-2M=1$ and it implies, by using (10), that $N-(k+l)-2m=N$ and so $k+l=0=m$. If $k+l=0$ then $i+j=N$. Finally we get $h_2^N(r,\rho,s)=r^N(b_2\sin Ns+c_2\cos Ns)$ and this proves statement (a).

Now we consider the case N odd. When $k+l+1-2M=1$ we obtain the same that in case N even, i.e., $r^N(b_2\sin Ns+c_2\cos Ns)$. When $k+l+1-2M=0$ for each M, from 0 to $(N-1)/2$, we obtain the terms $d_M^2 r^{N-2M-1}\rho^{2M+1}$. So in this case we get

$$h_2^N(r,\rho,s)=r^N(b_2\sin Ns+c_2\cos Ns)+\sum_{M=0}^{\frac{N-1}{2}} d_M^2 r^{N-2M-1}\rho^{2M+1},$$

and this shows statement (b). □

Lemma 4.3: *The following statements hold.*

(a) If N is even then

$$h_3(r,\rho,s)=a_3+r^{N-2}\rho(b_3\sin Ns+c_3\cos Ns)+r^N\rho^{-1}(d_3\sin Ns+e_3\cos Ns).$$

(b) If N is odd then

$$\begin{aligned} h_3(r,\rho,s)=a_3&+r^{N-2}\rho(b_3\sin Ns+c_3\cos Ns)+\\ &r^N\rho^{-1}(d_3\sin Ns+e_3\cos Ns)+\sum_{M=0}^{\frac{N-1}{2}} d_M^3 r^{N-2M-1}\rho^{2M},\end{aligned}$$

where a_3, b_3, c_3, d_3, e_3 and d_M^3's depend on the coefficients of the perturbation.

Proof: We have $H_3=H_3^1+H_3^N$ where

$$\begin{aligned} H_3^1 &= \tfrac{1}{N\rho}\Big(F_1^4\cos\big(N(\theta+s)\big)-F_1^3\sin\big(N(\theta+s)\big)\Big)-\tfrac{1}{r}\Big(F_1^2\cos\theta-F_1^1\sin\theta\Big),\\ H_3^N &= \tfrac{1}{N\rho}\Big(F_N^4\cos\big(N(\theta+s)\big)-F_N^3\sin\big(N(\theta+s)\big)\Big)-\tfrac{1}{r}\Big(F_N^2\cos\theta-F_N^1\sin\theta\Big).\end{aligned}$$

Applying Theorem 2.1 to system (7) and using the same arguments of Lemmas 4.1 and 4.2 we get

$$h_3^1(r,\rho,s)=\frac{1}{2\pi}\int_0^{2\pi}H_3^1(\theta,r,\rho,s)d\theta=\frac{a_{0010}^4-a_{0001}^3}{2N}-\frac{a_{1000}^2-a_{0100}^1}{2}.$$

Now we calculate $h_3^N(r,\rho,s)=\frac{1}{2\pi}\int_0^{2\pi}H_3^N(\theta,r,\rho,s)d\theta$. In a similar way to Lemmas 4.1 and 4.2 we get two sums of the form

$$\begin{aligned} h_3^N(r,\rho,s)=&\sum_{i+j+k+l=N} r^{i+j}\rho^{k+l-1}\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{[\frac{i+j}{2}]}\sum_{M=0}^{[\frac{k+l+1}{2}]} C_{mM}^{ijkl}(\theta)d\theta+\\ &\sum_{i+j+k+l=N} r^{i+j-1}\rho^{k+l}\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{[\frac{i+j+1}{2}]}\sum_{M=0}^{[\frac{k+l}{2}]} E_{mM}^{ijkl}(\theta)d\theta,\end{aligned}$$

where $C^{ijkl}_{mM}(\theta)$ is

$$c^{ijkl}_{mM}\cos\Big(((i+j-2m)\theta)\pm((k+l+1-2M)N(\theta+s))\Big)+$$
$$d^{ijkl}_{mM}\sin\Big(((i+j-2m)\theta)\pm((k+l+1-2M)N(\theta+s))\Big),$$

and $E^{ijkl}_{mM}(\theta)$ is

$$e^{ijkl}_{mM}\cos\Big(((i+j+1-2m)\theta\pm((k+l-2M)N(\theta+s))\Big)+$$
$$f^{ijkl}_{mM}\sin\Big(((i+j+1-2m)\theta)\pm((k+l-2M)N(\theta+s))\Big).$$

The terms which the integrals are non necessarily zero are given by

$$i+j-2m=N(k+l+1-2M) \tag{11}$$

in the first summand and by

$$i+j+1-2m=N(k+l-2M) \tag{12}$$

in the second summand.

The same arguments used in Lemmas 4.1 and 4.2 show that if N is even then the terms that remain are $r^{N-2}\rho(b_3\sin Ns+c_3\cos Ns)$ in the first summand and $r^N\rho^{-1}(d_3\sin Ns+e_3\cos Ns)$ in the second summand. If N is odd additionally to the terms obtained when N is even the first summand has terms of the form $r^{N-2M+1}\rho^{2M-2}$ with M from 1 to $(N+1)/2$, and the second summand has terms of the form $r^{N-2M-1}\rho^{2M}$ with M from 0 to $(N-1)/2$. □

Lemma 4.4: *Let N,α, and β be non–negative integers such that $\alpha+\beta=N$.*

(a) $$\int_0^{2\pi}\cos^\alpha t\sin^\beta t\cos(N(t+s))dt=\begin{cases}\frac{(-1)^{\beta/2}\pi}{2^{N-1}}\cos(Ns) & \text{if } \beta \text{ even,}\\ \frac{(-1)^{(\beta+1)/2}\pi}{2^{N-1}}\sin(Ns) & \text{if } \beta \text{ odd.}\end{cases}$$

(b) $$\int_0^{2\pi}\cos^\alpha t\sin^\beta t\sin(N(t+s))dt=\begin{cases}\frac{(-1)^{\beta/2}\pi}{2^{N-1}}\sin(Ns) & \text{if } \beta \text{ even,}\\ -\frac{(-1)^{(\beta+1)/2}\pi}{2^{N-1}}\cos(Ns) & \text{if } \beta \text{ odd.}\end{cases}$$

Proof: The expression $\cos^\alpha t\sin^\beta t$ may be written as

$$\cos^\alpha t\sin^\beta t\cos(N(t+s))=\left(\frac{e^{it}+e^{-it}}{2}\right)^\alpha\left(\frac{e^{it}-e^{-it}}{2i}\right)^\beta\cos(N(t+s)).$$

In the expansion of the right hand side we have to consider only terms e^{it} and e^{-it} such that have the highest degree, i.e., $\alpha+\beta=N$, because the integral of the other terms on the interval $[0,2\pi]$ are zero. So we get

$$\frac{1}{2^{N-1}}\left(\frac{e^{iNt}+(-1)^\beta e^{-iNt}}{2i^\beta}\right)=\begin{cases}\frac{(-1)^{\beta/2}}{2^{N-1}}\cos(Nt) & \text{if } \beta \text{ even,}\\ \frac{(-1)^{(\beta-1)/2}}{2^{N-1}}\sin(Nt) & \text{if } \beta \text{ odd.}\end{cases}$$

As $\cos(N(t+s))=\cos(Nt)\cos(Ns)-\sin(Nt)\sin(Ns)$, it follows statement (a).

Now defining $I(s) = \int_0^{2\pi} \cos^\alpha t \sin^\beta t \cos(N(t+s))dt$, we obtain that

$$\int_0^{2\pi} \cos^\alpha t \sin^\beta t \sin(N(t+s))dt = \frac{-I'(s)}{N}.$$

Hence statement (b) follows from statement (a). □

Lemma 4.5: *If N is even then the function h_3 of Lemma 4.3 is such that $b_3 = -c_1$, $c_3 = -b_1$, $d_3 = -c_2/N$ and $e_3 = b_2/N$.*

Proof: In order to simplify the proof, let $a^1_{ijkl} x_1^i x_2^j x_3^k x_4^l$ be a monomial of F_N^1 such that $i + j = N - 1$, $k = 0$ and $l = 1$. When we compute the expressions of h_1 and h_3, then this monomial appears in h_1 as

$$\frac{1}{2\pi} \int_0^{2\pi} a^1_{ijkl} \cos^{i+1}\theta \sin^j \theta \sin(N(\theta + s))d\theta, \tag{13}$$

and in h_3 as

$$\frac{1}{2\pi} \int_0^{2\pi} a^1_{ijkl} \cos^{i}\theta \sin^{j+1} \theta \sin(N(\theta + s))d\theta. \tag{14}$$

If $k = 0$ and $l = 1$ then, by Lemma 4.4, we have that (13) is equal to $(-1)^{j/2} a^1_{ijkl}$ $\sin(Ns)/2^N$ if j even, and to $-(-1)^{(j+1)/2} a^1_{ijkl} \cos(Ns)2^N$ if j odd, and (14) is equal to $(-1)^{(j+1)/2} a^1_{ijkl} sin(Ns)/2^N$ if $j + 1$ even, and to $-(-1)^{(j+2)/2} a^1_{ijkl} \cos(Ns)/2^N$ if $j + 1$ odd.

For j even the coefficient of the monomial appears in a sum that determines the coefficient of $r^{N-1}\rho \cos(Ns)$ in h_1, and also appears in a sum that determines the coefficient of $r^{N-2}\rho \sin Ns$ in h_3 with the opposite sign. In a similar way for j odd the coefficient of the monomial appears in a sum that determines the coefficient of $r^{N-1}\rho \sin Ns$ in h_1, and appears in a sum that determines the coefficient of $r^{N-2}\rho \cos Ns$ in h_3 with the same sign.

We can do the same for all monomials of F_N^2, F_N^3 and F_N^4, and easily check that $b_3 = -c_1$, $c_3 = -b_1$, $d_3 = -c_2/N$ and $e_3 = b_2/N$. □

5. Case *N* even

In this section we shall prove Theorem 1.1.

Proposition 5.1: *If N is even then we have*

$$\begin{aligned} h_1 &= h_1(r, \rho, s) = a_1 r + r^{N-1}\rho\big(b_1 \sin(Ns) + c_1 \cos(Ns)\big), \\ h_2 &= h_2(r, \rho, s) = a_2 \rho + r^N \big(b_2 \sin(Ns) + c_2 \cos(Ns)\big), \\ h_3 &= h_3(r, \rho, s) = a_3 + r^{N-2}\rho\big(- c_1 \sin(Ns) + b_1 \cos(Ns)\big) + \\ &\qquad r^N \rho^{-1}\big(- Nc_2 \sin(Ns) + Nb_2 \cos(Ns)\big). \end{aligned} \tag{15}$$

Proof: It follows from Lemmas 4.1, 4.2, 4.3 and 4.5. □

Proof: [Proof of Theorem 1.1] According to Proposition 5.1, the functions h_1, h_2 and h_3 are given by (15). We call $r^{N-1} = B$, $\rho/r = A$, $\sin Ns = z$ and $\cos Ns = w$.

After this change of variables we get

$$\begin{aligned}
\widetilde{h_1}(A,B,z,w) &= h_1/r = & a_1 + AB(b_1z + c_1w),\\
\widetilde{h_2}(A,B,z,w) &= h_2/r = & a_2A + B(b_2z + c_2w),\\
\widetilde{h_3}(A,B,z,w) &= \rho h_3/r = & a_3A + BA^2(-c_1z + b_1w) + B(-Nc_2z + Nb_2w),\\
\widetilde{h_4}(A,B,z,w) &= & z^2 + w^2 - 1.
\end{aligned}$$

Let $\widetilde{h_i} = \widetilde{h_i}(A,B,z,w)$ for $i = 1,2,3,4$. Now we solve $(\widetilde{h_1},\widetilde{h_2},\widetilde{h_3},\widetilde{h_4}) = (0,0,0,0)$. From $\widetilde{h_2} = 0$ we obtain

$$B = -\frac{Aa_2}{b_2z + c_2w}.$$

Substituting B in $\widetilde{h_1} = 0$, we obtain

$$A = \sqrt{\frac{a_1(b_2z + c_2w)}{a_2(b_1z + c_1w)}}, \quad \text{and so} \quad B = -\frac{1}{(b_2z + c_2w)}\sqrt{\frac{a_1(b_2z + c_2w)}{a_2(b_1z + c_1w)}}a_2.$$

Now substituting A and B in $\widetilde{h_3} = 0$ we obtain

$$\frac{B_1z^2 + B_2zw + B_3w^2}{(b_1z + c_1w)(b_2z + c_2w)} = 0, \tag{16}$$

where

$$\begin{aligned}
B_1 &= a_3b_1b_2 + a_1b_2c_1 + a_2b_1c_2N,\\
B_2 &= -a_1b_1b_2 + a_3b_2c_1 + a_3b_1c_2 + a_1c_1c_2 - a_2b_1b_2N + a_2c_1c_2N,\\
B_3 &= -a_1b_1c_2 + a_3c_1c_2 - a_2b_2c_1N.
\end{aligned}$$

The zeros of (16) are just $w = z = 0$, or a pair of crossing straight lines passing through the origin. So the maximum number of zeros of (16) and $z^2+w^2 = 1$ is four. Observe that for each zero (A,B,z,w) of $(\widetilde{h_1},\widetilde{h_2},\widetilde{h_3},\widetilde{h_4}) = (0,0,0,0)$, with $A > 0$ and $B > 0$, we can find N zeros (r,ρ,s) of $(h_1,h_2,h_3) = (0,0,0)$. In our case N is even, so the zeros (r,ρ,s) obtained from (A,B,z,w) are the same ones obtained from $(A,B,-z,-w)$. This completes the proof that the maximum number of zeros of $(h_1,h_2,h_3) = (0,0,0)$ is $2N$. So, by Theorem 2.1, the maximum number of limit cycle obtained via averaging theory for system (2) is $2N$. This proves statement (a).

Now we show that system (3) has $2N$ limit cycle. Computing h_1, h_2 and h_3 for this system we obtain

$$\begin{aligned}
h_1(r,\rho,s) &= -2\sin(\sqrt{2}\pi)r + r^{N-1}\rho\big(\sin Ns + 2\sin(\sqrt{2}\pi - Ns)\big),\\
h_2(r,\rho,s) &= \tfrac{2}{N}\Big(\sin(\sqrt{2}\pi)\rho - r^N\big(2\sin Ns + \sin(\sqrt{2}\pi - Ns)\big)\Big),\\
h_3(r,\rho,s) &= 3 + r^{N-2}\rho\big(\cos Ns - 2\cos(\sqrt{2}\pi - Ns)\big)+\\
&\qquad r^N\rho^{-1}\big(-4\cos Ns + 2\cos(\sqrt{2}\pi - Ns)\big).
\end{aligned}$$

The zeros of $(h_1, h_2, h_3) = (0, 0, 0)$ are

$$(r, \rho, s) = \left(1, 1, k\frac{2\pi}{N}\right) \text{ for } k \in \{0, 1, \ldots, N-1\},$$

and

$$(r, \rho, s) = \left(1, 2, \sqrt{2}\pi + k\frac{2\pi}{N}\right) \text{ for } k \in \{0, 1, \ldots, N-1\}.$$

The Jacobian determinant of $h = (h_1, h_2, h_3)$ computed at $\left(1, 1, k\frac{2\pi}{N}\right)$ for $k \in \{0, 1, \ldots, N-1\}$ is

$$-12(N-1)\Big(-5 + 4\cos\left(\sqrt{2}\pi\right)\Big)\sin\left(\sqrt{2}\pi\right) \neq 0,$$

and computed at $\left(1, 2, \sqrt{2}\pi + k\frac{2\pi}{N}\right)$ for $k \in \{0, 1, \ldots, N-1\}$ is

$$12(N-1)\Big(-5 + 4\cos\left(\sqrt{2}\pi\right)\Big)\sin\left(\sqrt{2}\pi\right) \neq 0.$$

Applying Theorem 2.1, the proof of statement (b) is done. □

6. Case N odd

In this section we shall prove Theorem 1.2.

Proposition 6.1: *If N is odd then we have*

$$\begin{aligned}
h_1(r, \rho, s) &= a_1 r + r^{N-1}\rho\big(b_1\sin(Ns) + c_1\cos(Ns)\big) + \sum_{M=0}^{\frac{N-1}{2}} d_M^1 r^{N-2M}\rho^{2M},\\
h_2(r, \rho, s) &= a_2\rho + r^N\big(b_2\sin(Ns) + c_2\cos(Ns)\big) + \sum_{M=0}^{\frac{N-1}{2}} d_M^2 r^{N-2M-1}\rho^{2M+1},\\
h_3(r, \rho, s) &= a_3 + r^{N-2}\rho\big(-c_1\sin(Ns) + b_1\cos(Ns)\big) +\\
&\qquad r^N\rho^{-1}\big(-Nc_2\sin(Ns) + Nb_2\cos(Ns)\big) + \sum_{M=0}^{\frac{N-1}{2}} d_M^3 r^{N-2M-1}\rho^{2M}.
\end{aligned}$$

Proof: It follows from Lemmas 4.1, 4.2, 4.3 and 4.5. □

Proof: [Proof of Theorem 1.2] In Proposition 6.1, the functions h_1, h_2 and h_3 are given. Now we perform the change $r^{N-1} = B$, $\rho/r = A$, $\sin Ns = z$ and $\cos Ns = w$. The functions $\widetilde{h_1} = h_1/r$, $\widetilde{h_2} = h_2/r$, $\widetilde{h_3} = \rho h_1/r$ in the new variables are

$$\begin{aligned}
\widetilde{h_1} &= a_1 + AB(b_1 z + c_1 w) + BP_1(A^2),\\
\widetilde{h_2} &= a_2 A + NB(b_2 z + c_2 w) + ABP_2(A^2),\\
\widetilde{h_3} &= a_3 A + BA^2(-c_1 z + b_1 w) + B(-c_2 z + b_2 w) + ABP_3(A^2),\\
\widetilde{h_4} &= z^2 + w^2 - 1,
\end{aligned}$$

where

$$P_i(A^2) = \sum_{M=0}^{\frac{N-1}{2}} d_M^i A^{2M}, \quad \text{for } i = 1, 2, 3.$$

We solve $\left(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}\right) = (0,0,0)$ and find a solution $B = B(A^2)$, $z = AZ(A^2)$, $w = AW(A^2)$, where $B(u)$ is the quotient of one polynomial of degree 2 by a polynomial of degree $(N+3)/2$, and $Z(u)$ and $W(u)$ are the quotient of one polynomial of degree $(N+1)/2$ by a polynomial of degree 2. Substituting z and w in the equation $\widetilde{h_4} = 0$, we obtain the quotient of one polynomial of degree $N+2$ in the variable A^2 by a polynomial of degree 4 in A^2. So the maximum number of positive roots A of the numerator of $\widetilde{h_4}$ is $N+2$. For each solution A_0 we have at most one $B_0 = B(A_0) > 0$ and just one pair $(z_0, w_0) = \big(z(A_0), w(A_0)\big)$. For each pair (z_0, w_0) we can find $s_1, \ldots, s_N \in [0, 2\pi)$ such that $\sin Ns_i = z_0$ and $\cos Ns_i = w_0$ for $i = 1, \ldots, N$. So, by Theorem 2.1, the maximum number of limit cycles obtained via averaging theory for system (2) is $N(N+2)$. This completes the proof of statement (a).

In order to prove that the previous bound $N(N+2)$ is attained for $N = 3$ we should prove that for each positive zero of $\widetilde{h_4}(A^2)$ the corresponding $B(A)$ is also positive. To ensure this last condition we impose that $B(A) = 1 + \alpha A^4$ with $\alpha > 0$. Then choosing a good collection of parameters,

$$\begin{aligned}
(a_1, b_1, c_1, d_0^1, d_1^1) &= \left(\frac{100}{59}, 0, -\frac{13}{16}, -\frac{100}{59}, 0\right),\\
(a_2, b_2, c_2, d_0^2, d_1^2) &= \left(-\frac{119}{18}, \frac{16}{13}, -\frac{59}{131}, \frac{4}{5}, 0\right),\\
(a_3, d_0^3, d_1^3) &= \left(\frac{161473}{113184}, -\frac{2821}{3930}, -\frac{169}{960}\right),
\end{aligned}$$

we obtain

$$B(A) = 1 + \frac{3441240088741}{6886432512000} A^2,$$

and

$$\begin{aligned}
\widetilde{h}_4 = {} & \frac{683524442249}{76441190400} A^{10} - \frac{106370960423}{8344829952} A^8 + \frac{161706458081946O5}{237833390702592} A^6 - \\
& \frac{923151210666125}{5500736967168} A^4 + \frac{169693234403855580625}{898133515906476096} A - \frac{22333852858522500000}{29279429820253561},
\end{aligned}$$

which has 5 simple positive zeros each one contained in a different interval of $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$ and $[4,5]$.

The Jacobian of $(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3})$ for our concrete case, and considering as a function of A, is of the form $AP_6(A^2)/P_2(A^2)$ and the numerator has no common zeros with $\widetilde{h}_4$ for that reason we can ensure that the 15 zeros are simple zeros of $(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3})$. This proves statement (b) of Theorem 1.2 for $N = 3$.

The cases $N = 5, 7, 9$ can be proved using the same arguments choosing a good selection of the parameters. $\square$

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References

[1] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128**, 2004, pp. 7–22.
[2] C. Christopher and C. Li, *Limit cycles of differential equations*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2007.
[3] Yu. S. Ilyashenko, *Centennial history of Hilbert's 16th problem*, Bull. Amer. Math. Soc. **39**, 2002, pp. 301–354.
[4] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13**, 2003, pp. 47–106.
[5] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.
[6] J. Murdock, *Normal Forms and Unfoldings for Local Dynamical Systems*, Springer Monographs in Math., New York, 2003.