Center boundaries for planar piecewise-smooth differential equations with two zones

Claudio A. Buzzi[a], Rubens Pazim[b], Set Pérez–González[a]

[a] *Department of Mathematics, IBILCE - UNESP Univ Estadual Paulista, Rua Cristovão Colombo 2265, Jardim Nazareth, CEP 15.054–000, Sao José de Rio Preto, SP, Brazil*
buzzi@ibilce.unesp.br and setperez@mat.uab.cat

[b] *Inst. de Ciências Naturais Humanas e Sociais, UFMT-Sinop, Av. Alexandre Ferronato 1.200, Setor Industrial, CEP 78.557–267, Sinop, MT, Brazil*
pazim@ufmt.br

Abstract

This paper is concerned with 1-parameter families of periodic solutions of piecewise smooth planar vector fields, when they behave like a center of smooth vector fields. We are interested in finding a separation boundary for a given pair of smooth systems in such a way that the discontinuous system, formed by the pair of smooth systems, has a continuum of periodic orbits. In this case we call the separation boundary as a *center boundary*. We prove that given a pair of systems that share a hyperbolic focus singularity p_0, with the same orientation and opposite stability, and a ray Σ_0 with endpoint at the singularity p_0, we can find a smooth manifold Ω such that $\Sigma_0 \cup \{p_0\} \cup \Omega$ is a center boundary. The maximum number of such manifolds satisfying these conditions is five. Moreover, this upper bound is reached.

Keywords:
Piecewise linear differential system, limit cycle, non-smooth differential system

1. Introduction

One of the most challenging problems in the qualitative theory of planar ordinary differential equations is the second part of the classical 16th Hilbert problem: the determination of an upper bound for the number of limit cycles for the class of polynomial vector fields of degree n. This problem remains unsolved if $n \geq 2$. The case $n = 1$ has a trivial answer because we can not have limit cycles for linear systems. By the other hand, we can have limit cycles for planar piecewise linear differential systems. It means that this problem, in the context of piecewise smooth systems, has attracted much attention.

The study of piecewise linear differential systems goes back to Andronov and coworkers [1]. These systems are used to model many real processes and different modern devices, see for more details [2] and the references therein.

The simplest case of piecewise linear differential systems is the one in which we have two half-planes separated by a straight line W. If both linear vector fields coincide at each point $w \in W$ we say that it is the case of continuous piecewise linear differential systems. In 1990, Lum and Chua conjectured that a continuous piecewise linear vector field in the plane with two zones has at most one limit cycle, see [14]. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [8].

In the literature we can find a lot of works that deal with limit cycles of discontinuous piecewise linear differential systems, see for instance [5, 6, 9, 10, 11, 12]. Han and Zang, in [10], provide discontinuous systems with two limit cycles, and they conjecture that the maximum number of limit cycles for this class is exactly two. However, in [11], Huan and Yang presented numerical analysis showing that an example with three limit cycles could exists. Later on, Llibre and Ponce provide in [12] a proof of the existence of such three limit cycles. In [5] the authors obtain three limit cycles from a piecewise perturbation of a linear center, and they can choose from which periodic orbits of the linear center the limit cycles bifurcate. To the best of our knowledge, we do not know an example of planar piecewise linear systems separated by a straight line W with four or more limit cycles.

In planar piecewise linear differential systems, the separation boundary W between the two zones plays an important role. After using some broken line as the boundary between linear zones, Braga and Mello in [3] put in evidence the important role of the separation boundary in determining the number of limit cycles. In [3] the authors exhibit an example with seven limit cycles having W as a polygonal curve and they state the conjecture: "*Given $n \in \mathbb{N}$ there is a piecewise linear system with two zones in the plane with exactly n limit cycles*". This conjecture was proved by the same authors in the paper [4]. The main idea of [13] for three zones is still valid when only two zones exist. Novaes and Ponce in [15] gave another solution to the Braga–Mello Conjecture.

In the paper [4], the authors consider piecewise linear systems sharing a singular point of focus type, both in the Jordan Normal Form. In Lemma 2 of [4] is proved that there exists a piecewise linear separation boundary such that the discontinuous system has a center. Our work is inspired in this result. We are interested in find separation boundary for a given pair of piecewise smooth systems in such a way that the discontinuous system has a continuum of periodic solutions. In this case we call the separation boundary as a *center boundary*. In [15] the center boundary considered is the y-axis, this choice is possibly due to the special nature of the eigenvalues of the system chosen. Here in this work we discuss the case of piecewise linear systems when we have two foci not necessarily in the Jordan Normal Form. We deal not only with piecewise linear systems but also with piecewise smooth systems not necessarily linear. Our work is also related with stability issues in the active field of switched control systems, see for instance [16, 17, 19].

We consider pairs of differential systems of class $\mathcal{C}^r$, $r \geq 2$, in the following way. Let $U \subset \mathbb{R}^2$ be an open set and consider $f_1, f_2 : U \to \mathbb{R}^2$ of class $\mathcal{C}^r$. We denote the pair of differential systems

$$\dot{X} = f_1(X), \tag{1}$$

and

$$\dot{X} = f_2(X), \tag{2}$$

by $Z = (f_1, f_2)$. The set of all pairs $Z = (f_1, f_2)$ of systems (1) and (2) we denote by $\mathfrak{X}^r$. For each $X_0 \in U$, $i = 1, 2$, we denote the solution of $\dot{X} = f_i(X)$ that passes through X_0 at $t = 0$ by γ_i, i.e., $\gamma_i(0, X_0) = X_0$.

Let $W \subset U$ be a piecewise smooth manifold in such a way that the set $U \setminus W$ has two connected components, i.e., $U = U_1 \cup U_2 \cup W$, where U_1 and U_2 are connected open sets. Given the pair $Z = (f_1, f_2)$ we define $Z_W = (f_1, f_2, W)$ as a piecewise smooth differential system

$$\dot{X} = \begin{cases} f_1(X) \text{ if } X \in U_1 \cup W, \\ f_2(X) \text{ if } X \in U_2 \cup W. \end{cases} \tag{3}$$

System (3) is accept to be multi-valued at W.

Let $\Sigma \subset U$ be a 1-dimensional manifold transversal to both vector fields, (1) and (2). It means that $f\mathrm{n}_i(X) := f_i(X) \cdot \mathrm{n}(X) \neq 0, \forall X \in \Sigma,$ and $i = 1, 2$, with $\cdot$ the inner product and $\mathrm{n}(X)$ the normal vector to Σ at the point X. We say that Σ is a *cross-section* for $Z = (f_1, f_2)$ if $f\mathrm{n}_1(X) f\mathrm{n}_2(X) > 0$ and a *slide-section* for $Z = (f_1, f_2)$ if $f\mathrm{n}_1(X) f\mathrm{n}_2(X) < 0$. In the case that

$f\mathrm{n}_1(X)f\mathrm{n}_2(X) < 0$ we can define the *Fillipov vector field* on Σ. The definition and basic results of this theory can be found in [7]. In the present paper we treat only cases that $f\mathrm{n}_1(X)f\mathrm{n}_2(X) > 0$.

Now we define the meaning of *periodic orbit* for Z_W, given by (3). Assume that there are points $p_1, p_2 \in W$, positive times $t_1, t_2 \in \mathbb{R}$ and solutions γ_1 of (1), satisfying $p_1 = \gamma_1(0, p_1)$, and γ_2 of (2), satisfying $p_2 = \gamma_2(0, p_2)$. We say that $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{\gamma_i(t, p_i),\ 0 \leq t \leq t_i\}$, is a periodic orbit for Z_W if $p_2 = \gamma_1(t_1, p_1)$, $p_1 = \gamma_2(t_2, p_2)$ and $\Gamma_i \subset U_i$. In the case that we have a continuum of periodic orbits for Z_W we say that W is a *center boundary* for Z_W (See Figure 1).

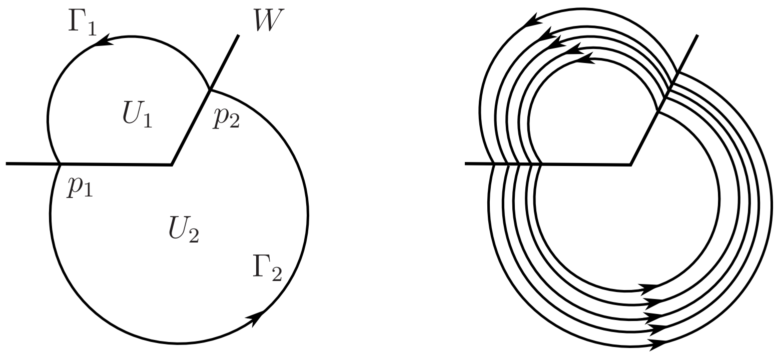

Figure 1: Periodic Orbits of $Z = (f_1, f_2)$.

Given a pair $Z = (f_1, f_2) \in \mathfrak{X}^r$, we want to build a piecewise smooth manifold W in such a way that W is a center boundary for system $Z_W = (f_1, f_2, W)$. Through the paper we present some hypotheses about the pair Z and the cross-section Σ such that the construction is possible. Another question is, for a fixed pair $Z = (f_1, f_2) \in \mathfrak{X}^r$ and a cross-section Σ, to determine the maximum number of center boundaries that contains Σ. The main result in the paper that answers these questions is the following.

Theorem 1. *Consider $Z = (f_1, f_2) \in \mathfrak{X}^r$ such that f_1 and f_2 share a focus singularity at p_0, with the same orientation and opposite stability, and a ray Σ_0 with endpoint at p_0. There exists at least one piecewise smooth manifold W such that $W \supset \Sigma_0$ and W is a center boundary for Z_W. The maximum number of such manifolds W, satisfying $W \supset \Sigma_0$ and W is a center boundary for Z_W, is five. Moreover, this upper bound is reached (See Example 1).*

A more precise statement of this result is Theorem 3 in Section 4.

Example 1. *[Five Center Boundaries] Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be the pair of linear vector fields $f_1(X) = A_1X$ and $f_2(X) = A_2X$, where*

$$A_1 = \begin{pmatrix} -0.01 & -1 \\ 1 & -0.01 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0.5 & -0.05 \\ 20 & 0.5 \end{pmatrix}.$$

So, there are five Center Boundaries $W_i = \Sigma_0 \cup \Omega_i$, where Ω_i, $i = 1, \ldots, 5$ are given by:

$$\begin{aligned}
\Omega_1 &= \{(x, y) \in \mathbb{R}^2, y = -134.8775564625664\ldots x,\ x > 0\};\\
\Omega_2 &= \{(x, y) \in \mathbb{R}^2, y = -21.3368256507493\ldots x,\ \ x > 0\};\\
\Omega_3 &= \{(x, y) \in \mathbb{R}^2, y = \ \ 8.31842273952277\ldots x,\ \ x > 0\ \};\\
\Omega_4 &= \{(x, y) \in \mathbb{R}^2, y = -1.781004127809957\ldots x,\ x < 0\};\ \textit{and},\\
\Omega_5 &= \{(x, y) \in \mathbb{R}^2, y = \ \ 1.544443235738568\ldots x,\ x < 0\}.
\end{aligned}$$

A geometric representation of Example 1 can be seen in Figure 2. Observe that any intersection of the red and the blue orbits defines in this case a linear switching boundary Ω_j, $j = 1, \ldots, 5$, represented by the brown straight lines. Each one of the Ω_j together with the half $y < 0$ of the vertical axis, denoted by Σ_0, specifies a piecewise-linear system with two zones such that all non-equilibrium orbits are closed, i.e. it is a center. In Section 3 we give more details about Example 1. After finding the center boundary, our system becomes stable, but not asymptotically stable. In [19] and [16] the authors consider one pair of systems unstable and determine a separation boundary with four zones, in order to stabilize it asymptotically. Such construction is intensely used in control theory.

This paper is organized as follows. In section 2 we present concepts and prove some lemmas, they are crucial to the proof of the main results. In Section 3 we deal with the linear version of Theorem 1, in Section 4 we state and prove the Theorem 3, which is a more precise statement of Theorem 1 and in Section 5 We analyze a specific RLC Electrical Circuit subordinated to a periodic switching law, connecting the theory studied in the previous sections with the control theory.

2. Technical Results

Definition 1 (Hypotheses H and HL)**.** *Given a pair $Z = (f_1, f_2) \in \mathfrak{X}^r$. We say that Z satisfies the Hypotheses H if there exists p_0 a singular point of both systems, such that the linear part of f_1 and f_2 at p_0 are $A_1 = (a^1_{ij})_{i,j=1,2}$ and $A_2 = (a^2_{ij})_{i,j=1,2}$, respectively, and the following statements hold:*

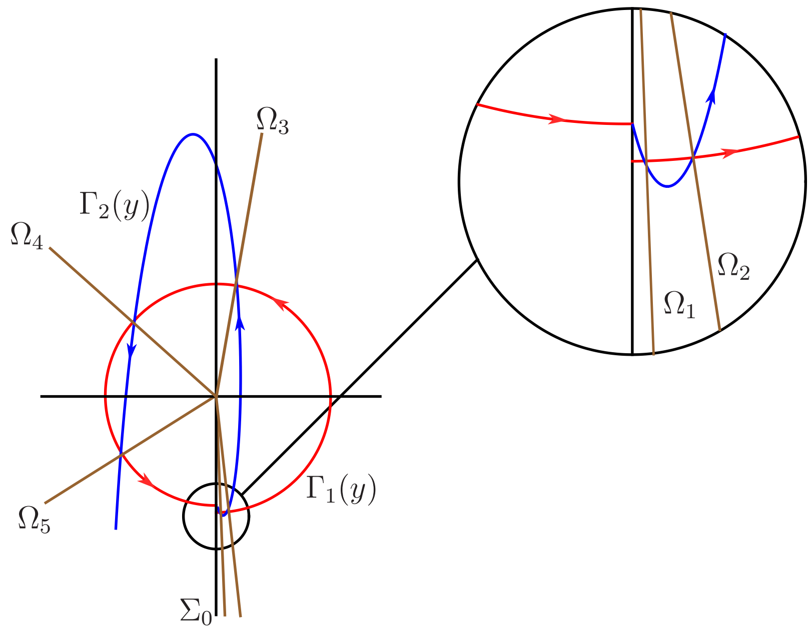

Figure 2: Five Center Boundaries. The red line is an orbit of vector field $\dot{X} = A_1 X$ starting at point $(0, -1)$ for negative time; The blue line is an orbit of vector field $\dot{X} = A_2 X$ starting at point $(0, -1)$ for positive time; and, the brown lines represent the five Center Boundaries Ω_i, $i = 1, \ldots, 5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(i) $\mathrm{tr}(\mathrm{A}_1) < 0$, $\mathrm{tr}(\mathrm{A}_2) > 0$;

(ii) $\Delta_1 = (\mathrm{tr}(\mathrm{A}_1))^2 - 4\det(\mathrm{A}_1) < 0$; $\Delta_2 = (\mathrm{tr}(\mathrm{A}_2))^2 - 4\det(\mathrm{A}_2) < 0$;

(iii) $a_{21}^1 > 0$, $a_{21}^2 > 0$*; and,*

(iv) $\mathrm{Hess}(\det(f_1(X), f_2(X)), p_0) \neq 0$.

Where $\mathrm{Hess}(g, q)$ *is the determinant of the Hessian matrix of function* g *at point* q*.*

If f_1 *and* f_2 *are linear systems then we say that the pair satisfies the Hypotheses HL when (i), (ii) and (iii) are verified.*

In Definition 1, the statements (i)–(iv) are required in order that: (i) the singular points of systems f_1 and f_2 have opposite stability, (ii) they are of foci type, (iii) both foci are in anti-clockwise orientation, and (iv) we can apply the Morse's Theorem in some technical results.

Lemma 1. *If $Z = (f_1, f_2) \in \mathfrak{X}^r$ satisfies the Hypotheses H (resp. HL) and Σ is a ray with endpoint at the common singular point of f_1 and f_2 then there exists a change of coordinates that translates the singular point to origin and transforms the linear part of f_1 at this point in its Jordan Form and Σ at $\Sigma_0 = \{(x, y) \in \mathbb{R}^2, x = 0$ and $y < 0\}$. Moreover, in the new coordinates, the Hypotheses H (resp. HL) are still satisfied.*

Proof. First we apply a translation that sends p_0 to the origin. It does not change the linear part of both systems and does not change the Hessian of the function $\det(f_1(X), f_2(X))$. Now we apply in both systems a linear change of coordinates J that sends the linear part of system f_1 to its Jordan Normal Form. It is obvious that a linear change of coordinates J does not change trace, determinant and Hessian of the function $\det(f_1(X), f_2(X))$. Finally we apply a rotation to sends $J(\Sigma)$ to $\Sigma_0 = \{(x, y) \in \mathbb{R}^2, x = 0$ and $y < 0\}$, where $J(\Sigma)$ is the image of Σ by J. It is also clear that a rotation is a linear change of coordinates that keeps the Jordan Normal Form of a hyperbolic focus singularity. So, statements $(i) - (iv)$ of Definition 1 still hold. □

From now on, we will consider that a pair Z satisfying Hypotheses H or HL has its common singular point at the origin and the linear part of f_1 is in its Jordan canonical form, i.e., $a^1_{11} = a^1_{22}$ and $a^1_{12} = -a^1_{21}$. So, having $Z = (f_1, f_2) \in \mathfrak{X}^r$ satisfying Hypotheses H, the vector fields (1) and (2) are written

$$\dot{X} = A_1 X + \tilde{f}_1(X), \text{ and} \tag{4}$$

$$\dot{X} = A_2 X + \tilde{f}_2(X), \tag{5}$$

where $\tilde{f}_i(0) = D\tilde{f}_i(0) = 0$, $i = 1, 2$. In addition, we will consider the vector field $f_{1\varepsilon}$ obtained as a linear perturbation of (4) as follows

$$\dot{X} = f_{1\varepsilon}(X) = (A_1 + \varepsilon Id)X + \tilde{f}_1(X). \tag{6}$$

Notation 1. *Given a point $(0, y) \in \Sigma_0$, we denote by $\gamma_1(t, (0, y))$, $\gamma_2(t, (0, y))$ and $\gamma_{1\varepsilon}(t, (0, y))$ the orbits passing through $(0, y)$ defined by (4), (5) and (6) respectively and satisfying $\gamma_1(0, (0, y)) = \gamma_2(0, (0, y)) = \gamma_{1\varepsilon}(0, (0, y)) = (0, y)$. Let us consider that $T_1(y)$, $T_2(y)$ and $T_{1\varepsilon}(y)$ are the positive time that $\gamma_1(-t, (0, y))$, $\gamma_2(t, (0, y))$ and $\gamma_{1\varepsilon}(-t, (0, y))$ spend until they return to Σ_0, respectively. We consider also the following definitions:*

(i) $\Gamma_1(y) = \{\gamma_1(-t, (0, y)), 0 < t < T_1(y)\}$;

(ii) $\Gamma_2(y) = \{\gamma_2(t, (0, y)), 0 < t < T_2(y)\}$*; and,*

(iii) $\Gamma_{1\varepsilon}(y) = \{\gamma_{1\varepsilon}(-t, (0, y)), 0 < t < T_{1\varepsilon}(y)\}$.

A geometric representation of orbits $\Gamma_1(y)$ and $\Gamma_2(y)$ can be seen in Figure 3.

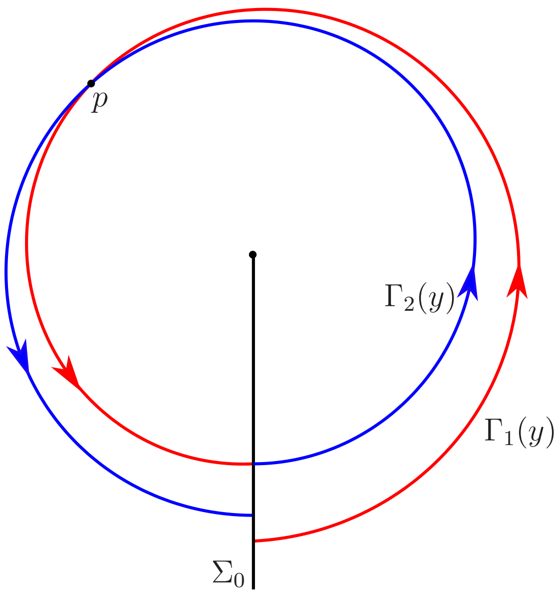

Figure 3: Orbits $\Gamma_1(y)$ and $\Gamma_2(y)$ presented in Notation 1; red $\Gamma_1(y)$ and blue $\Gamma_2(y)$.

As we will see later, the curves $\Gamma_1(y)$ and $\Gamma_{1\varepsilon}(y)$ should intersects the curve $\Gamma_2(y)$. At the intersection point the curves can be either transversal or tangent. In the transversal case we say that the contact order between the curves is one, and in the tangent case it is bigger or equal to two. The following two definitions detail each case.

Definition 2. *Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, and $\alpha, \beta : I \to \mathbb{R}^2$ be two smooth curves, given by $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and $\beta(t) = (\beta_1(t), \beta_2(t))$, such that $\alpha(0) = \beta(0) = p \in \mathbb{R}^2$, $\alpha'(0) \neq (0, 0)$ and $\beta'(0) \neq (0, 0)$:*

- *if* $\alpha_1'(0) = 0$ *and* $\beta_1'(0) \neq 0$ *then we say that* α *and* β *have a contact of order one at* p*;*

- *if* $\alpha_1'(0) \neq 0$ *and* $\beta_1'(0) = 0$ *then we say that* α *and* β *have a contact of order one at* p*;*

- *if* $\alpha_1'(0) \neq 0$ *and* $\beta_1'(0) \neq 0$*, then by Implicit Function Theorem there exists an unique smooth function* $t_1 = t_1(x)$ *such that* $\alpha_1(t_1(x)) - \alpha_1(0) = x$ *for all* x *in a neighborhood of* $x = 0$*; and there exists an unique smooth function* $t_2 = t_2(x)$ *such that* $\beta_1(t_2(x)) - \beta_1(0) = x$ *for all* x *in a neighborhood of* $x = 0$*. Let* f_α *and* f_β *be real smooth functions defined by* $f_\alpha(x) = \alpha_2(t_1(x)) - \alpha_2(0)$ *and* $f_\beta(x) = \beta_2(t_2(x)) - \beta_2(0)$*. We say that* α *and* β *have a contact of order* n *at* p *if* $0 = f_\alpha'(0) - f_\beta'(0) = \cdots = f_\alpha^{(n-1)}(0) - f_\beta^{(n-1)}(0)$ *and* $f_\alpha^{(n)}(0) - f_\beta^{(n)}(0) \neq 0$*; and,*

- *if* $\alpha_1'(0) = \beta_1'(0) = 0$ *then* $\alpha_2'(0) \neq 0$ *and* $\beta_2'(0) \neq 0$*. Then by Implicit Function Theorem there exists an unique smooth function* $t_1 = t_1(y)$ *such that* $\alpha_2(t_1(y)) - \alpha_2(0) = y$ *for all* y *in a neighborhood of* $y = 0$*; and there exists an unique smooth function* $t_2 = t_2(y)$ *such that* $\beta_2(t_2(y)) - \beta_2(0) = y$ *for all* y *in a neighborhood of* $y = 0$*. Let* f_α *and* f_β *be real smooth functions defined by* $f_\alpha(y) = \alpha_1(t_1(y)) - \alpha_1(0)$ *and* $f_\beta(y) = \beta_1(t_2(y)) - \beta_1(0)$*. We say that* α *and* β *have a contact of order* n *at* p *if* $0 = f_\alpha'(0) - f_\beta'(0) = \cdots = f_\alpha^{(n-1)}(0) - f_\beta^{(n-1)}(0)$ *and* $f_\alpha^{(n)}(0) - f_\beta^{(n)}(0) \neq 0$*.*

Definition 3. *Let us suppose that* $\Gamma_1(y)$ *intersects* $\Gamma_2(y)$ *at* $p \in \mathbb{R}^2$*. We say that* $\Gamma_1(y)$ *and* $\Gamma_2(y)$ *have:*

(i) *a transversal contact at* p *if* $\Gamma_1(y)$ *and* $\Gamma_2(y)$ *have a contact of order one at* p*;*

(ii) *a crossing tangential contact at* p *if* $\Gamma_1(y)$ *and* $\Gamma_2(y)$ *have a contact of order* n*, with* n *odd and bigger than one, at* p*; and,*

(iii) *a non-crossing tangential contact at* p *if* $\Gamma_1(y)$ *and* $\Gamma_2(y)$ *have a contact of order* n*, with* n *even, at* p*.*

Consider the notation of Definition 2. In case (iii), we say that it is an outer tangential contact *at* p *if* $f_{\Gamma_1(y)}^{(n)}(0) - f_{\Gamma_2(y)}^{(n)}(0) > 0$ *(See Figure 4-(a)), and we* *say that it is an* inner tangential contact *at p if $f^{(n)}_{\Gamma_1(y)}(0) - f^{(n)}_{\Gamma_2(y)}(0) < 0$ (See Figure 4-(b)). In case (ii), we say that it is a* crossing inn tangential contact *at p if $f^{(n)}_{\Gamma_1(y)}(0) - f^{(n)}_{\Gamma_2(y)}(0) < 0$ (See Figure 4-(c)), and we say that it is a* crossing out tangential contact *at p if $f^{(n)}_{\Gamma_1(y)}(0) - f^{(n)}_{\Gamma_2(y)}(0) > 0$ (See Figure 4-(d)). In case (i), we say that it is a* transversal inn contact *at p if $\det(f_1(p), f_2(p)) > 0$ (See, for instance, point p_2 in Figure 4-(a)), and we say that it is a* transversal out contact *at p if $\det(f_1(p), f_2(p)) < 0$ (See, for instance, point p_1 in Figure 4-(a)).*

Remark 1. *In all cases of Figure 4 we have that Γ_1 is an orbit of* (1), *Γ_2 is an orbit of* (2) *and Γ_1 and Γ_2 have a tangential contact point at p. Dashed lines are also orbits of* (2) *close to Γ_2. Assume that Γ_1 is given by $\alpha(t)$, such that $\alpha(0) = p$. We consider $p_1 = \alpha(t_1)$ and $p_2 = \alpha(t_2)$, with $t_1 < 0 < t_2$, arbitrary points in the orbit Γ_1. In Figure 4 we are denoting $v_{ij} = v_i(p_j)$, for $i, j = 1, 2$. Here f_1 and f_2 are the vector fields that define systems* (1) *and* (2). *Observe that $\det(v_{11}, v_{21})$ is positive in the pictures (b) and (c), and negative in the pictures (a) and (d). Observe also that $\det(v_{12}, v_{22})$ is positive in the pictures (a) and (c), and negative in the pictures (b) and (d). In other words we can characterize the tangential contact points just analyzing the behavior of the function $d(t) = \det(f_1(\alpha(t)), f_2(\alpha(t)))$. If $d(t)$ passes:*

- *from negative to positive then Γ_1 and Γ_2 have an outer tangential contact at p (See Figure 4-(a));*
- *from positive to negative then Γ_1 and Γ_2 have an inner tangential contact at p (See Figure 4-(b));*
- *from positive to zero and back to positive then Γ_1 and Γ_2 have a crossing inn tangential contact at p (See Figure 4-(c)); and,*
- *from negative to zero and back to negative then Γ_1 and Γ_2 have a crossing out tangential contact at p (See Figure 4-(d)).*

The tangential (transversal) contact point p of $\Gamma_1(y)$ with $\Gamma_2(y)$ occur if, and only if, $f_1(p)$ and $f_2(p)$ are parallel (transversal). In the lemmas presented in sequel, we will carefully explore these issues.

Lemma 2. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of functions and $\mathcal{D}$ be the region of the plane $\{X \in \mathbb{R}^2 : \det(f_1(X), f_2(X)) = 0\}$. The following statements hold:*

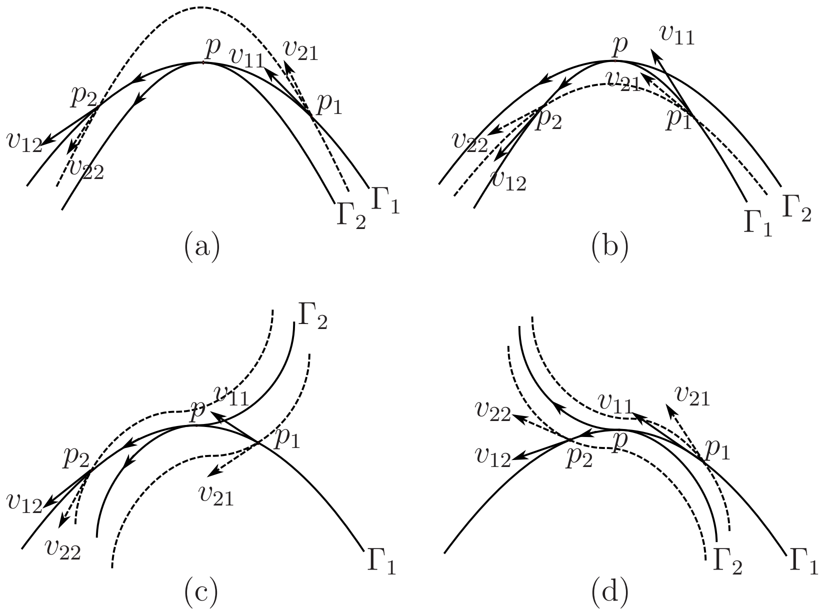

Figure 4: Tangential Contact Points.

(i) if $Z = (f_1, f_2) \in \mathfrak{X}^r$ satisfies the Hypotheses HL, then $\mathcal{D}$ can be composed by a single or a pair of straight lines both containing the origin or it only contains the origin; and,

(ii) if $Z = (f_1, f_2) \in \mathfrak{X}^r$ satisfies the Hypotheses H, then in a neighborhood U of the origin, $\mathcal{D} \cap U$ is composed by two regular curves with their intersection just being the origin or it only contains the origin.

Proof. (i) In linear case, we have that

$$\det(f_1(X), f_2(X)) = \det(A_1X, A_2X) = \alpha x^2 + \beta xy + \gamma y^2$$

where

$$\begin{aligned}
\alpha &= a_{11}^1 a_{21}^2 - a_{21}^1 a_{11}^2 \\
\beta &= a_{11}^1 a_{22}^2 + a_{12}^1 a_{21}^2 - a_{21}^1 a_{12}^2 - a_{22}^1 a_{11}^2 \\
\gamma &= a_{12}^1 a_{22}^2 - a_{22}^1 a_{12}^2.
\end{aligned}$$

Therefore we obtain the following relation,

$$H_d := \mathrm{Hess}(\det(A_1X, A_2X), 0) = 4\alpha\gamma - \beta^2,$$

where $\text{Hess}(g, q)$ is the determinant of the Hessian matrix of function g at point q. H_d is the discriminant of $\det(A_1X, A_2X)/y^2$ with respect of x/y changed of sign. So, the aspect of $\mathcal{D}$ depends on the sign of H_d. Such that, if $H_d < 0$, $\mathcal{D}$ will be defined by two real straight lines crossing each other at the origin. If $H_d = 0$, $\mathcal{D}$ contains one straight line passing through the origin, twice. And, finally, if $H_d > 0$, the equation $\det(A_1X, A_2X) = 0$ defines a pair of complex straight lines and the origin will be the unique real point contained in $\mathcal{D}$.

(ii) As determinant is a bilinear function we have

$$Hess(\det(f_1(X), f_2(X)), 0) = Hess(\det(A_1X, A_2X), 0) = H_d.$$

As $H_d \neq 0$, we can apply Morse's Lemma to our function $\hat{f} = \det(f_1(X), f_2(X))$ around the origin. Therefore there exists a diffeomorphism $h : U \to h(U)$ in U, neighborhood of the origin, such that $\hat{f}(h(X)) = \det(A_1X, A_2X)$ for every $X \in U$. From the first part of this lemma, the proof is completed.

□

Let us denote $\mathcal{R} = (\mathcal{D} \cup \Sigma_0) \cap U$, where U is the neighborhood of the origin given by Lemma 2. For the case $\tilde{f_1} \equiv \tilde{f_2} \equiv 0$, it means f_1 and f_2 are linear systems, we have that the neighborhood U is the whole Cartesian plane. The set $\mathcal{R}$ divides U in at most 5 regions. Each one of these regions is called *sector* and the common border of two consecutive sectors is called *ray*.

Remark 2. *Note that in the proof of Lemma 2 the configuration of $\mathcal{D}$ depends on the signal of H_d. We will give a more complete analysis:*

(i) if $H_d > 0$ then the origin is an unique point contained in $\mathcal{D}$. Consequently $\det(f_1(X), f_2(X)) \neq 0$ *and we have only transversal contact points of $\Gamma_1(y)$ and $\Gamma_2(y)$;*

(ii) if $H_d = 0$ then f_1 and f_2 are linear, $\mathcal{D}$ is composed by a single straight line containing the origin and the sign of the function $\det(f_1(X), f_2(X))$ *is either non-negative or non-positive. By Remark 1 we do not have non-crossing tangential contact points of $\Gamma_1(y)$ and $\Gamma_2(y)$; and,*

(iii) if $H_d < 0$ then $\mathcal{D} \cap U$ is composed by two regular curves with their intersection just being the origin. And in this case we will prove in Lemma 5 that $\Gamma_1(y)$ and $\Gamma_2(y)$ do not have crossing tangential contact points.

Lemma 3. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of functions under Hypotheses H or HL and consider $\Gamma_1(y)$ and $\Gamma_2(y)$ given by Notation 1. If $(0, y) \in U \cap \Sigma_0$ then, about the tangential and transversal contact points of $\Gamma_1(y)$ with $\Gamma_2(y)$ we affirm:*

(i) the tangential contact points occur only in $\mathcal{D} \cap U$; and,

(ii) the transversal contact points occur inside sectors determined by $\mathcal{R}$ and there exists at most one transversal contact point in each sector.

Proof. We start proving (i). We know that X belongs to $\mathcal{D} \cap U$ if and only if $\det(f_1(X), f_2(X)) = 0$, i.e. $f_1(X)$ and $f_2(X)$ are parallel and so the statement (i) is proved.

For (ii), it is clear that transversal contact points occur inside sectors determined by $\mathcal{R}$. Assume that two of them, we named q_1 and q_2, occur in the same sector determined by $\mathcal{R}$. For sure one of them is a transversal inn contact point and the other one is a transversal out contact point. Thus $\det(f_1(q_1), f_2(q_1)) \cdot \det(f_1(q_2), f_2(q_2)) < 0$ and it follows from Intermediate Value Theorem the existence of a point q_0 in the same sector such that $\det(f_1(q_0), f_2(q_0)) = 0$. This contradiction finishes the proof of statement (ii). $\square$

Lemma 4. *Let g_1 and g_2 be smooth vector fields defined on $\mathbb{R}^2$ and $p \in \mathbb{R}^2$ such that $g_1(p) \neq 0$, $g_2(p) \neq 0$ and $\nabla \det(g_1(p), g_2(p)) \neq 0$. Assume that $\gamma_1(t)$ and $\gamma_2(t)$ are solutions of g_1 and g_2, respectively, such that $\gamma_1(0) = \gamma_2(0) = p$, $\gamma_1'(0)$ is parallel to $\gamma_2'(0)$ and γ_1 crosses γ_2 at the point p. Then γ_1 and γ_2 are tangent to the curve $\det(g_1(X), g_2(X)) = 0$ at the point p.*

Proof. Applying the Flow Box Theorem to the vector field g_1 in a neighborhood V of p we can find a real smooth function $f : V \to \mathbb{R}$ such that the orbits of g_1 coincide with the level curves of f and the gradient vector of f, denoted by ∇f, is equal to the orthogonal vector field of g_1, denoted by $g_1^{\perp}$. We can assume that $f(\gamma_1(t)) = 0$ for all t. Consider the function

$g(t) = f(\gamma_2(t))$. So we have

$$\begin{aligned} g'(t) &= \langle \nabla f(\gamma_2(t)), \gamma_2'(t)\rangle = \langle g_1^{\perp}(\gamma_2(t)), g_2(\gamma_2(t))\rangle \\ &= \det(g_1(\gamma_2(t)), g_2(\gamma_2(t))). \end{aligned} \tag{7}$$

Thus $g(0) = f(p) = 0$ and $g'(0) = \det(g_1(p), g_2(p)) = 0$. Note if $g''(0) < 0$ then $t = 0$ is a local maximum for the function g. It implies that $f(\gamma_2(t)) \leq 0$ in a neighborhood of $t = 0$, and so in this situation $\gamma_2(t)$ does not cross $\gamma_1(t)$ because $f(\gamma_1(t)) \equiv 0$. In that same way we can not have $g''(0) > 0$. It implies that $g''(0) = 0$. Differentiating (7) we obtain

$$\begin{aligned} g''(t) &= \langle \nabla \det(g_1(\gamma_2(t)), g_2(\gamma_2(t))), \gamma_2'(t)\rangle \\ &= \langle \nabla \det(g_1(\gamma_2(t)), g_2(\gamma_2(t))), g_2(\gamma_2(t))\rangle. \end{aligned}$$

At $t = 0$ we have $\langle \nabla \det(g_1(p), g_2(p)), g_2(p)\rangle = 0$. This implies that the orbit of g_2 is tangent to the curve $\det(g_1(X), g_2(X)) = 0$ at the point p. □

Lemma 5. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of functions under Hypotheses H and consider $\Gamma_1(y)$ and $\Gamma_2(y)$ given by Notation 1. Then $\Gamma_1(y)$ and $\Gamma_2(y)$ do not have a crossing tangential contact point.*

Proof. By Hypotheses H, $\mathrm{Hess}(\hat{f}(X), 0) \neq 0$. Then, in a neighborhood U of the origin, the set $\mathcal{D}$ is composed by the single point $(0, 0)$ or a pair of regular curves passing through the point $(0, 0)$. The proof of case $\mathcal{D} = \{(0, 0)\}$ is straightforward and will be omitted. In the case $\mathcal{D} \neq \{(0, 0)\}$, let $p \neq (0, 0)$ belonging to $\mathcal{D}$; it is clear that $\nabla \det(f_1(p), f_2(p)) \neq 0$. Applying Lemma 4 we get that $f_1(p)$ is a vector tangent to the curve contained in $\mathcal{D}$ at the point p.

Now we consider Morse's Diffeomorphism $h : U \to h(U)$ given by Lemma 2 and the systems $\overline{f_i} = Dh^{-1} \circ f_i \circ h$, $i = 1, 2$. Since $h(X) = X + \mathcal{O}(|X|^2)$, it follows that $D\overline{f_1}(0) = A_1$; as h is diffeomorphism, $\overline{f_1}(h^{-1}(p))$ is a vector parallel to a ray, contained in $\hat{\mathcal{D}} = \{X \in \mathbb{R}^2 : \det(A_1(X), A_2(X)) = 0\}$, at the point $h^{-1}(p)$. This is a contradiction, effectively in polar coordinates, the angular component of system $\overline{f_1}$ is of the form $\dot{\theta} = a_{21}^1 + \mathcal{O}(\theta, r)$, with $a_{21}^1 > 0$, and the rays in $\hat{\mathcal{D}}$ are half straight lines.

For case f_i linear, i=1,2; it is enough to consider $U = R^2$ and $h(X) = X$.

□

Lemma 6. *[Bifurcation of Tangential Points] Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of functions under Hypotheses H or HL. Consider $\Gamma_1(y)$, $\Gamma_{1\varepsilon}(y)$ and*

$\Gamma_2(y)$ given by Notation 1. Assume that for all $(0,y) \in U \cap \Sigma_0$ the orbits $\Gamma_1(y)$ and $\Gamma_2(y)$ have n inner tangential contact points, m outer tangential contact points and k transversal contact points. Then, for sufficiently small $|\varepsilon|$, the orbits $\Gamma_{1\varepsilon}(y)$ and $\Gamma_2(y)$ have:

(i) $2m + k$ *contact points if* $\varepsilon < 0$;

(ii) $2n + k$ *contact points if* $\varepsilon > 0$;

and all of them are transversal contact points.

Proof. In polar coordinates systems (4), (5) and (6), respectively, are given by:

$$\begin{cases} \dot{r} = a_{11}^1 r + f_{11}(\theta, r), \\ \dot{\theta} = a_{12}^1 + f_{12}(\theta, r), \end{cases}$$

$$\begin{cases} \dot{r} = f_{21}(\theta, r), \\ \dot{\theta} = f_{22}(\theta, r), \end{cases}$$

and

$$\begin{cases} \dot{r} = (a_{11}^1 + \varepsilon) r + f_{11}(\theta, r), \\ \dot{\theta} = a_{12}^1 + f_{12}(\theta, r), \end{cases}$$

where $f_{11}(\theta, 0) = f_{12}(\theta, 0) = \dfrac{\partial f_{11}}{\partial r}(\theta, 0) = 0$ for all θ. Let $p = (\theta_0, r_0)$ be an outer tangential contact point of $\Gamma_1(y)$ and $\Gamma_2(y)$. Denote by $r_1(\theta)$, $r_2(\theta)$ and $r_{1\varepsilon}(\theta)$, respectively, the solutions of the initial value problems

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{a_{11}^1 r + f_{11}(\theta, r)}{a_{12}^1 + f_{12}(\theta, r)}, \\ r(-\frac{\pi}{2}) = y, \end{cases}$$

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{f_{21}(\theta, r)}{f_{22}(\theta, r)}, \\ r(-\frac{\pi}{2}) = y \end{cases}$$

and

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{(a_{11}^1 + \varepsilon) r + f_{11}(\theta, r)}{a_{12}^1 + f_{12}(\theta, r)}, \\ r(-\frac{\pi}{2}) = y. \end{cases}$$

We affirm, if $s > -\frac{\pi}{2}$, $\varepsilon < 0$ and $|y|$ sufficiently small, then $r_{1\varepsilon}(s) < r_1(s)$. Effectively

$$\begin{aligned} r_{1\varepsilon}(s) + y &= \int_{-\frac{\pi}{2}}^{s} \frac{(a_{11}^1 + \varepsilon)r + f_{11}(\theta, r)}{a_{12}^1 + f_{12}(\theta, r)} d\theta \\ &< \int_{-\frac{\pi}{2}}^{s} \frac{a_{11}^1 r + f_{11}(\theta, r)}{a_{12}^1 + f_{12}(\theta, r)} d\theta = r_1(s) + y. \end{aligned}$$

Since $p = (\theta_0, r_0)$ is an outer tangential contact point, we can see clearly that $r_1(\theta_0) = r_2(\theta_0)$ and for sufficiently small $\rho > 0$ we have $r_1(\theta) > r_2(\theta)$ for all $\theta \in [\theta_0 - \rho, \theta_0 + \rho] \setminus \{\theta_0\}$. There exists $\varepsilon_0 < 0$ such that for all ε, with $\varepsilon_0 < \varepsilon < 0$, we have

$$r_1(\theta_0 - \rho) > r_{1\varepsilon}(\theta_0 - \rho) > r_2(\theta_0 - \rho)$$

and

$$r_1(\theta_0 + \rho) > r_{1\varepsilon}(\theta_0 + \rho) > r_2(\theta_0 + \rho).$$

By the other hand, the fact that $r_{1\varepsilon}(\theta_0) < r_1(\theta_0) = r_2(\theta_0)$ and the Intermediate Value Theorem imply that there exist $\theta_1 \in (\theta_0 - \rho, \theta_0)$ and $\theta_2 \in (\theta_0, \theta_0 + \rho)$ such that $r_{1\varepsilon}(\theta_1) = r_2(\theta_1)$ and $r_{1\varepsilon}(\theta_2) = r_2(\theta_2)$. According to Lemma 3 these two points $p_1 = (\theta_1, r_2(\theta_1))$ and $p_2 = (\theta_2, r_2(\theta_2))$ are transversal contact points. If $\varepsilon > 0$ then $r_{1\varepsilon}(\theta) > r_1(\theta) \geq r_2(\theta)$ for all $\theta \in [\theta_0 - \rho, \theta_0 + \rho]$. Thus, for θ in this interval we can say that there are not contact points. The fact that $|\varepsilon|$ is arbitrarily small imply that the transversal contact points are kept. So, each outer tangential contact point of $\Gamma_1(y)$ and $\Gamma_2(y)$ becomes two transversal contact points for $\varepsilon < 0$ and disappears for $\varepsilon > 0$. In a similar way we can prove that each inner tangential contact point of $\Gamma_1(y)$ and $\Gamma_2(y)$ becomes two transversal contact points for $\varepsilon > 0$ and disappears for $\varepsilon < 0$. This concludes the proof. □

Lemma 7. *Under the same conditions of Lemma 3, let n_1, n_e and n_o be the numbers of transversal, non-crossing tangential and crossing tangential contact points of $\Gamma_1(y)$ and $\Gamma_2(y)$ respectively. Then $n_e \cdot n_o = 0$ and $n_1 + 2n_e + n_o \leq 5$.*

Proof. The proof of the lemma is a consequence of the analysis of the sign of the H_d which was made in the Remark 2. The Following three results are sufficient for to prove the lemma:

(i) $n_e \cdot n_o = 0$;

(ii) if $n_o \neq 0$ then $n_o + n_1 \leq 5$; and,

(iii) if $n_e \neq 0$ then $2n_e + n_1 \leq 5$.

We start proving (i), if $H_d = 0$ ($H_d \neq 0$) then, by Remark 2, $n_e = 0$ ($n_o = 0$). Therefore $n_e \cdot n_o = 0$. Note that the analysis of case $H_d = 0$ is made with f_1 and f_2 being linear vector fields. This is necessary to satisfy the Hypotheses HL.

For (ii) if $n_o \neq 0$ by (i) $n_e = 0$, consequently $H_d = 0$ and f_i is linear, for $i = 1, 2$. By Remark 2 the set $\mathcal{D}$ divides U in two sectors and the set $\mathcal{R}$ divides U in at most three sectors. Thus from Lemma 3 it follows that $n_1 \leq 3$ and $n_0 \leq 2$. This completes the proof of (ii).

Finally for the proof of (iii), if $n_e \neq 0$ by (i) $n_o = 0$ and consequently $H_d \neq 0$. Thus we claim:

(a) if a ray $\mathcal{R}_i$ of $\mathcal{R}$ contains a non-crossing tangential contact point then the adjacent sectors of $\mathcal{R}_i$ do not posses transversal contact points; and,

(b) non-crossing tangential contact points do not occur in consecutive rays of $\mathcal{R}$.

In order to proof (a), let $p \in \mathcal{R}_i$ be a non-crossing tangential contact point and q a transversal contact point in a adjacent sector of the $\mathcal{R}_i$. By Lemma 6, for ε suitable, we can apply the linear perturbation (6) in system $\dot{X} = f_1(X)$ and consequently the point p bifurcates in two transversal contact points in a neighborhood of p and the transversal contact points persist in a neighborhood of q. Thus we have three transversal contact points in two adjacent sectors of perturb system, but by Lemma 3 this is a contradiction.

For the proof of (b) let p and q be non-crossing tangential contact points in consecutive rays $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ respectively, with $\mathcal{R}_i, \mathcal{R}_{i+1} \subset \mathcal{R}$. By (a) it follows that inside of adjacent sectors to $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ we do not have transversal contact points. Thus, p and q are both non-crossing tangential contact points of the same nature, outer or inner tangential contact points. By Lemma 6, for an ε suitable, we can apply the linear perturbation (6) in system $\dot{X} = f_1(X)$ and consequently the points p and q bifurcate in four transversal contact points belonging to three sectors determined by $\mathcal{R}$. By Lemma 3 this is contradiction and so the statement (b) is proved.

Since the number of sectors determined by $\mathcal{R}$ is at most five, it follows from statements (a) and (b) that $2n_e + n_1 \leq 5$. $\square$

Lemma 8. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of functions under Hypotheses H or HL, and consider $\Gamma_1(y)$ and $\Gamma_2(y)$ given by Notation 1. Then, for each $(0, y) \in U \cap \Sigma_0$ there exists a point on the plane where $\Gamma_1(y)$ intersects $\Gamma_2(y)$. Furthermore if the Hypotheses H are satisfied then the intersection is transversal.*

Proof. According to Lemma 1 we can consider $\Sigma_0 = \{(x, y) \in \mathbb{R}^2, x = 0 \text{ and } y < 0\}$. Let $\Pi_2 : \Sigma_0 \to \Sigma_0$ be the Poincaré Map given by system (5). For each point $(0, y) \in \Sigma_0$, with $y < 0$, we have that $\Pi_2(0, y) = (0, z_1)$, with $z_1 < y$. This is true because system (5) has a repealing focus. Consider $S(y) \subset \Sigma_0$ the segment with end point $(0, y)$ and $\Pi_2(0, y) = (0, z_1)$. It is obvious that $\Gamma(y) = \Gamma_2(y) \cup S(y)$ is a closed curve.

Let $\Pi_1 : \Sigma_0 \to \Sigma_0$ be the Poincaré Map given by system (4). Now using that system (4) has an attracting focus, we have that $\Pi_1^{-1}(0, y) = (0, z_2)$, where $z_2 < y$. Let $\tau_1 > 0$ be the time such that $\gamma_1(-\tau_1, (0, y)) = (0, z_2)$. It is obvious that for $\varepsilon > 0$, small enough, we have that $\gamma_1(-\varepsilon, (0, y))$ is in the interior of $\Gamma(y)$ and $\gamma_1(-\tau_1 + \varepsilon, (0, y))$ is in the exterior of $\Gamma(y)$. According to the definition of Poincaré Map Π_1, clearly, we can see that $\{\gamma_1(t, (0, y)),\ -\tau_1 + \varepsilon < t < -\varepsilon\}$ does not intersect the segment $S(y)$. So it implies that $\Gamma_1(y)$ intersects $\Gamma_2(y)$ in a point p distinct from $(0, y)$ (See Figure 3).

If the Hypotheses H are satisfied then by Lemma 5 we have that p is not a crossing tangential contact point. Since $\Gamma_1(y)$ crosses $\Gamma_1(y)$ in p we have that p is a transversal contact point. $\square$

3. Linear Case

In this section we present the linear version of the main result of the paper (Theorem 1), i.e. we consider the case when f_i, $i = 1, 2$, are linear vector fields.

Theorem 2. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of linear vector fields that satisfies the Hypotheses HL and let Σ_0 be a ray with endpoint at the common singular point p_0 of f_1 and f_2. Then it holds:*

(i) there exists a ray Ω, with endpoint p_0, such that $W = \Sigma_0 \cup \{p_0\} \cup \Omega$ is a center boundary for $Z_W = (f_1, f_2, W)$; and,

(ii) the upper bound for the number of rays Ω_i, with endpoint p_0, such that $W_i = \Sigma_0 \cup \{p_0\} \cup \Omega_i$ is a center boundary for $Z_{W_i} = (f_1, f_2, W_i)$, is five. Moreover, this upper bound is reached.

Proof. First of all, according to Lemma 1, we can consider $p_0 = (0,0)$, $\Sigma_0 = \{(0,y) \in \mathbb{R}^2, y < 0\}$ and assume that A_1 is in its Jordan Normal Form, where $f_1(X) = A_1X$ and $f_2(X) = A_2X$.

In order to prove statement (i), consider the $\mathcal{C}^\infty$-function $G : \mathbb{R}^2 \times \mathbb{R}_- \to \mathbb{R}^2$ defined by $G(t,s,y) = e^{A_2 s}(0,y) - e^{-A_1 t}(0,y)$, with $y < 0$. We want to solve

$$G(t,s,y) = 0. \tag{8}$$

By Lemma 8, for each $y < 0$ the equation (8) has at least a solution (τ_1, τ_2, y). We can divide (8) by $-y > 0$ and obtain

$$e^{A_2 s}(0,-1) - e^{-A_1 t}(0,-1) = 0. \tag{9}$$

So $\tau = (\tau_1, \tau_2)$ is a solution of (9). It means that the fly times τ_1 and τ_2 do not depend on the initial point $(0,y) \in \Sigma_0$. We know that the flow of a linear system, for a fixed time t, sends ray with endpoint $(0,0)$ to ray with endpoint $(0,0)$. In particular the image of the ray Σ_0 by $e^{-A_1\tau_1}$ and $e^{A_2\tau_2}$ coincide and so it is a ray Ω with endpoint $(0,0)$. So $W = \Sigma_0 \cup \{p_0\} \cup \Omega$ is a center boundary for $Z_W = (f_1, f_2, W)$.

Now we prove statement (ii). According to the construction of Ω in (i) we have that the center boundary is given by the intersection of $\Gamma_1(y)$ and $\Gamma_2(y)$, for $(0,y) \in \Sigma_0$. Lemma 7 ensures that for each $y < 0$, $\Gamma_1(y) \cap \Gamma_2(y)$ has at most five elements. It implies that we have at most five center boundaries of the form $W_i = \Sigma_0 \cup \{p_0\} \cup \Omega_i$. Finally, the Example 1 is a proof that this upper bound is reached. $\square$

Now we show the detail of Exemple 1 as follows.

Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be the pair of linear vector fields $f_1(X) = A_1X$ and $f_2(X) = A_2X$, where

$$A_1 = \begin{pmatrix} -0.01 & -1 \\ 1 & -0.01 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0.5 & -0.05 \\ 20 & 0.5 \end{pmatrix}.$$

So, there are five rays Ω_i, $i = 1, 2 \ldots, 5$, like in Theorem 2, and they are given by:

$$\begin{aligned}
\Omega_1 &= \{(x,y) \in \mathbb{R}^2, y = -134.8775564625664\ldots x,\ x > 0\};\\
\Omega_2 &= \{(x,y) \in \mathbb{R}^2, y = -21.3368256507493\ldots x,\quad x > 0\};\\
\Omega_3 &= \{(x,y) \in \mathbb{R}^2, y = \ \ 8.31842273952277\ldots x,\quad x > 0\ \};\\
\Omega_4 &= \{(x,y) \in \mathbb{R}^2, y = -1.781004127809957\ldots x,\ x < 0\};\text{ and,}\\
\Omega_5 &= \{(x,y) \in \mathbb{R}^2, y = \ \ 1.544443235738568\ldots x,\ x < 0\}.
\end{aligned}$$

In fact, solving numerically the equation (9) we obtain the solution:

$$\begin{aligned}
(\tau_{11}, \tau_{21}) &= (6.275771311120894\ldots, 0.1472099548970395\ldots);\\
(\tau_{12}, \tau_{22}) &= (6.236352249719108\ldots, 0.75306960711223\ldots);\\
(\tau_{13}, \tau_{23}) &= (3.261233618070344\ldots, 1.9649520341626272\ldots);\\
(\tau_{14}, \tau_{24}) &= (2.629977598472333\ldots, 4.623573047888529\ldots);\text{ and,}\\
(\tau_{15}, \tau_{25}) &= (0.5746034278947866\ldots, 4.789458190335897\ldots).
\end{aligned}$$

Thus, we denote $q_i = e^{-A_1\tau_{1i}}(0,-1) = e^{A_2\tau_{2i}}$ for $i = 1, \ldots, 5$, and we write

$$\begin{aligned}
q_1 &= (0.007894119577290202\ldots, -1.0647395648297642\ldots);\\
q_2 &= (0.0498285069740593\ldots, -1.0631821653064175\ldots);\\
q_3 &= (0.12331238264053886\ldots, 1.0257645288981185\ldots);\\
q_4 &= (-0.5026329580685281\ldots, 0.8951913733052014\ldots);\text{ and,}\\
q_5 &= (-0.5466339180793478\ldots, -0.8442450566431993\ldots).
\end{aligned}$$

Clearly the points q_i are in intersection of $\Gamma_1(-1)$ with $\Gamma_2(-1)$ and the rays Ω_i are passing through the points q_i, for $i = 1, \ldots, 5$.

4. Non Linear Case

In this section we deal with the non linear case. Next theorem is the most important result of the paper and in its proof some other results obtained in previous sections of the paper will be used.

Theorem 3. *Let $Z = (f_1, f_2) \in \mathfrak{X}^r$ be a pair of vector fields that satisfies the Hypotheses H and let Σ_0 be a ray with endpoint at the common singular point p_0 of f_1 and f_2. Then, in a neighborhood U of p_0, the following statements hold:*

(i) there exists an 1-dimensional manifold Ω, with endpoint p_0, such that $W = (\Sigma_0 \cup \{p_0\} \cup \Omega) \cap U$ is a center boundary for $Z_W\big|_U = (f_1, f_2, W)\big|_U$; and,

(ii) the upper bound for the number of manifolds Ω_i, with endpoint p_0, such that $W_i = (\Sigma_0 \cup \{p_0\} \cup \Omega_i) \cap U$ is a center boundary for $Z_{W_i}\big|_U = (f_1, f_2, W_i)\big|_U$, is five. Moreover, this upper bound is reached.

Proof. The proof is organized as follows. In the first part of the proof we will set the notation recalling some definitions given in the paper and stating new notations to be used in the proof. Basically we are using the convention that notations with hat is about linear systems and without hat is about nonlinear systems. In the second part of the proof we will find a curve Ω and a neighborhood U of the origin, such that Ω, the origin and Σ_0 are connected, i.e. $W = (\Sigma_0 \cup \{(0,0)\} \cup \Omega) \cap U$ is a center boundary for $Z_W\big|_U = (f_1, f_2, W)\big|_U$. Finally, in the third part of the proof we will show that the curve Ω is of class $\mathcal{C}^r$.

We start with the first part of the proof establishing some notations. As in Theorem 2, by Lemma 1, we can consider $p_0 = (0,0)$, $\Sigma_0 = \{(0,y) \in \mathbb{R}^2, y < 0\}$ and assume that A_1 is in its Jordan Normal Form. According to our notation, equations (4) and (5), we have that $f_i(X) = A_i X + \widetilde{f}_i(X)$, where $\widetilde{f}_i(0) = D\widetilde{f}_i(0) = 0$, for $i = 1, 2$. Consider $(\theta, r) = \psi(x, y)$ the polar change of coordinates and denote:

- $\gamma_i(t, (0, y_0))$ the solution of $\dot{X} = f_i(X)$ satisfying the initial condition $\gamma_i(0, (0, y_0)) = (0, y_0)$, $i = 1, 2$;
- $\hat{\gamma}_i(t, (0, y_0))$ the solution of $\dot{X} = A_i X$ satisfying $\hat{\gamma}_i(0, (0, y_0)) = (0, y_0)$, $i = 1, 2$;
- $\beta_i(t, (0, y_0)) = \psi \circ \gamma_i(t, (0, y_0))$ and $\hat{\beta}_i(t, (0, y_0)) = \psi \circ \hat{\gamma}_i(t, (0, y_0))$, $i = 1, 2$;
- $\hat{\Gamma}_1(y_0) = \left\{e^{-A_1 t}(0, y_0), 0 < t < 2\pi/a_{12}^1\right\}$;
- $\hat{\Gamma}_2(y_0) = \left\{e^{A_2 t}(0, y_0), 0 < t < 4\pi/\sqrt{\Delta_2}\right\}$;
- $\Lambda_i(y_0) = \psi(\Gamma_i(y_0))$ and $\hat{\Lambda}_i(y_0) = \psi(\hat{\Gamma}_i(y_0))$, $i = 1, 2$;
- $\mathcal{D} = \{X \in \mathbb{R}^2 : \det(f_1(X), f_2(X)) = 0\}$, $\mathcal{R} = \mathcal{D} \cup \Sigma_0$ and $\mathcal{S} = \psi(\mathcal{R})$; and,
- $\hat{\mathcal{D}} = \{X \in \mathbb{R}^2 : \det(A_1 X, A_2 X) = 0\}$, $\hat{\mathcal{R}} = \hat{\mathcal{D}} \cup \Sigma_0$ and $\hat{\mathcal{S}} = \psi(\hat{\mathcal{R}})$.

We know that the set $\hat{\mathcal{R}}$ consists of n rays, $1 \le n \le 5$, which we define $\hat{\mathcal{R}}_j$, $j = 1, \ldots, n$. By Morse's Lemma there is a diffeomorphism h, defined in a neighborhood U_0 of the origin, such that $h(\hat{\mathcal{D}} \cap U_0) = \mathcal{D} \cap h(U_0)$. Thus we have $\mathcal{R}_j = h(\hat{\mathcal{R}}_j)$ and we also denote $\hat{\mathcal{S}}_j = \psi(\hat{\mathcal{R}}_j) = \{\theta = \theta_j\}$ and $\mathcal{S}_j = \psi(\mathcal{R}_j)$, for $j = 1, \ldots, n$.

For the second part of the proof we want to obtain the curve Ω and the neighborhood U. The strategy to obtain Ω will be to find $\delta > 0$ and a parametrization $p : (-\delta, 0) \to \mathbb{R}^2$ such that $\Omega = \{p(y_0), 0 < -y_0 < \delta\}$ and $\lim_{y_0 \to 0} p(y_0) = 0$. In order to do that we observe that by Lemma 8 for each $y_0 < 0$ there is a point $\hat{p}(y_0)$ such that $\hat{\Gamma}_1(y_0)$ intersects $\hat{\Gamma}_2(y_0)$ transversely at $\hat{p}(y_0)$. It follows from Theorem 2 that there is a ray $\mathcal{R}^* \neq \hat{\mathcal{R}}_j$, for all $j \in \{1, \ldots, n\}$, such that $\hat{p}(y_0) \in \mathcal{R}^*$ for all $y_0 < 0$. Denote by $\mathcal{S}^* = \psi(\mathcal{R}^*) = \{\theta = \theta^*\}$. It is clear that there are $i, j \in \{1, \ldots, n\}$ such that $\theta_i < \theta^* < \theta_j$. For simplicity assume that $i = 1$ and $j = 2$, i.e. $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ are consecutive rays of $\hat{\mathcal{S}}$ such that $\theta_1 < \theta^* < \theta_2$ (if $n = 1$ then $\theta_2 = \theta_1 + 2\pi$).

Consider the strip $L_\rho = \{(\theta, r) : r < \rho\}$. We observe that for a given $\varepsilon_1 > 0$ there exists $\rho > 0$ such that, if $0 < r < \rho$ then $dist(\hat{\mathcal{S}}_j \cap L_r, \mathcal{S}_j \cap L_r) < \varepsilon_1$, $j = 1, \ldots, n$ and $\psi^{-1}(L_\rho) \subset U_0$. Take $U = \psi^{-1}(L_\rho)$, from the Theorem of Dependence on Initial Conditions and Parameters it follows that there exists $\delta_1 > 0$ such that

$$\text{if } |y_0| < \delta_1 \text{ then } \hat{\Lambda}_i(y_0) \subset L_\rho, \tag{10}$$

for $i = 1, 2$.

It is clear that systems $\dot{X} = A_i X$, under the Hypotheses H, are foci. Thus in polar coordinates, satisfy $\dot{\theta} \neq 0$. It follows that for each $y_0 < 0$ the sets $\hat{\Lambda}_i(y_0)$ are graphs of functions $r_i(\theta, y_0)$, $\quad i = 1, 2$. Thus, $\hat{q}(y_0) := \psi(\hat{p}(y_0)) = (\theta^*, r_1(\theta^*, y_0)) = (\theta^*, r_2(\theta^*, y_0))$. Now, let ε_1 be a real number that satisfies $0 < \varepsilon_1 < \min\{\theta_1 - \theta^*, \theta^* - \theta_2\}/3$ and consider the strip $F = \{(\theta, r), |\theta - \theta^*| < 2\varepsilon_1 \text{ and } r < \rho\}$. For a fixed $y_0 < 0$, denote by $\eta_i^\pm = \hat{\Lambda}_i(y_0) \cap \{\theta = \theta^* \pm 2\varepsilon_1\}$, i=1,2; and $\eta^\pm$ the midpoint between $\eta_1^\pm$ and $\eta_2^\pm$.

Take $\varepsilon_2 y_0 = -\min\{dis(\eta^\pm, \hat{\Lambda}_i(y_0)), i = 1, 2\}$. We state that there exists $\delta_2 > 0$ such that

$$\text{if } |y_0| < \delta_2 \text{ then } \|\hat{\beta}_i(t, (0, y_0)) - \beta_i(t, (0, y_0))\| < \varepsilon_2 |y_0|. \tag{11}$$

In fact,

$$\gamma_i(t, (0, y_0)) = \gamma_i(t, (0, 0)) + \frac{\partial}{\partial y_0}\left[\gamma_i(t, (0, y_0))\right]\Big|_{y_0=0} \cdot (0, y_0) + R_i(t, y_0)$$

where

$$\lim_{y_0\to 0}\frac{\|R_i(t,y_0)\|}{|y_0|}=0,\quad i=1,2.$$

Since $\gamma_i(t,(0,0))=0$ and by Remark 2, on page 84 of [18], we have

$$\frac{\partial}{\partial y_0}\left[\gamma_i(t,(0,y_0))\right]\Bigg|_{y_0=0}\cdot(0,y_0)=e^{A_i t}\cdot(0,y_0)=\hat{\gamma}_i(t,(0,y_0)),\quad i=1,2.$$

So, it follows that $\gamma_i(t,(0,y_0))-\hat{\gamma}_i(t,(0,y_0))=R_i(t,y_0)$, $i=1,2$. Note that ψ is a diffeomorphism, so the proof of (11) is completed.

Now consider $V_i(y_0)$ the tubular neighborhood of $\hat{\Lambda}_i(y_0)$ of radius ε_2 (See Figure 5). Denote by $\hat{\Lambda}_i^+(y_0)$ and $\hat{\Lambda}_i^-(y_0)$ the two regular curves which determine the border of $V_i(y_0)$. The fact that $\hat{\Gamma}_1(y_0)$ intersects $\hat{\Gamma}_2(y_0)$ transversely at $\hat{p}(y_0)$ implies $\hat{\Lambda}_1^\pm(y_0)$ intersect $\hat{\Lambda}_2^\pm(y_0)$ transversally in F and by the choice of ε_2 we have that $\hat{\Lambda}_1^\pm(y_0)$ intersect $\hat{\Lambda}_2^\mp(y_0)$ in F. Denote p_i, $i=1,\ldots,4$; the points given by $p_1=\hat{\Lambda}_1^+(y_0)\cap\hat{\Lambda}_2^+(y_0)$, $p_2=\hat{\Lambda}_1^-(y_0)\cap\hat{\Lambda}_2^+(y_0)$, $p_3=\hat{\Lambda}_1^-(y_0)\cap\hat{\Lambda}_2^-(y_0)$ and $p_4=\hat{\Lambda}_1^+(y_0)\cap\hat{\Lambda}_2^-(y_0)$ (See Figures 5 and 6).

Let K be the region determined by arcs $\widehat{p_1p_2}$, $\widehat{p_2p_3}$, $\widehat{p_3p_4}$ and $\widehat{p_4p_1}$ (See Figure 6). For $0<-y_0<\delta$ we have that $K\subset F$, and so by (10) K is contained in one sector determined by $\mathcal{S}$. By equation (11), the orbit $\Lambda_1(y_0)$ does not cross the arcs $\widehat{p_4p_1}$ and $\widehat{p_2p_3}$ and the orbit $\Lambda_2(y_0)$ does not cross the arcs $\widehat{p_1p_2}$ and $\widehat{p_3p_4}$. Since $\Lambda_1(y_0)$ crosses $\widehat{p_1p_2}$ and $\widehat{p_3p_4}$ and $\Lambda_2(y_0)$ crosses $\widehat{p_2p_3}$ and $\widehat{p_4p_1}$ we have that $\Lambda_1(y_0)$ intersects $\Lambda_2(y_0)$ in K and the intersection point $q(y_0)=\psi(p(y_0))$ is unique.

Take $\Omega=\{p(y_0),0<-y_0<\delta\}$. As

$$\lim_{y_0\to 0}\max\{\|X\|,\ \text{such that } X\in\Gamma_i(y_0)\}=0,\quad i=1 \text{ or } i=2,$$

it follows that $\lim_{y_0\to 0}p(y_0)=0$. Thus we have that Ω is connected to the origin. So the second part of the proof is finished.

The last part of the proof consists in to prove the differentiability of Ω. We observe that the intersection of $\Gamma_1(y_0)$ with $\Gamma_2(y_0)$ in K is transversal, because $K\cap\mathcal{D}=\emptyset$. Let s_0 and t_0 be the corresponding fly times for the intersection, i.e., $\gamma_2(s_0,(0,y_0))=\gamma_1(-t_0,(0,y_0))=p(y_0)$. Consider the equation

$$\Phi(t,s,y)=0, \tag{12}$$

where $\Phi:\mathbb{R}^2\times\mathbb{R}_-\to\mathbb{R}^2$ is the $\mathcal{C}^r$-function defined by $\Phi(t,s,y)=\gamma_2(s,(0,y))-\gamma_1(-t,(0,y))$, with $y<0$. We observe that

$$\det(\partial_{(s,t)}\Phi(t_0,s_0,y_0))=\det(f_1(p(y_0)),f_2(p(y_0)))\neq 0. \tag{13}$$

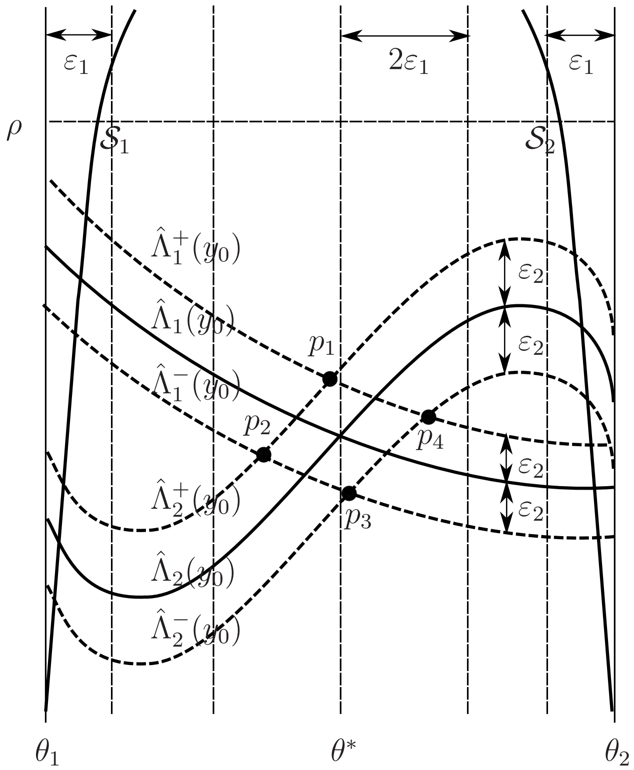

Figure 5: Construction of Tubular neighborhoods.

So, by Implicit Function Theorem, there are $\mathcal{C}^r$-functions $s = s(y)$ and $t = t(y)$, defined in an open interval $I_0 \ni y_0$, such that $\Phi(t(y), s(y), y) \equiv 0$. We have that Ω, in a neighborhood of $p(y_0)$, is given by

$$\Omega = \{\gamma_1(-t(y), (0, y)) : y \in I_0\}.$$

The fact that Ω is image of an open interval by a differentiable map implies that Ω is differentiable at $p(y_0)$. By arbitrariness of y_0 it follows that Ω is differentiable.

The proof of (ii) is similar to that given in proof of Theorem 2.

□

Note that the condition (iv) in Definition 1 is only technical. We observe that all pair of linear systems satisfying conditions (i), (ii) and (iii) of Definition 1 has a Center Boundary (See Theorem 2). For example, considering

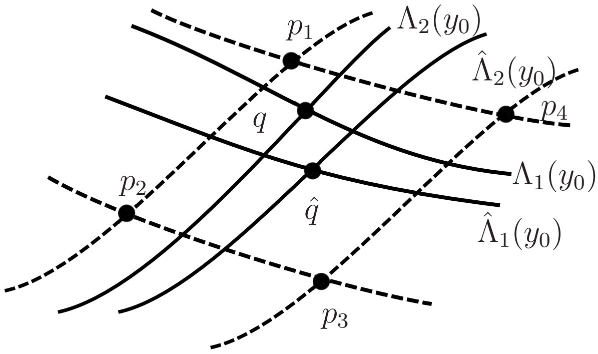

Figure 6: Transversal intersection in the tubular neighborhoods.

$f_1(X) = A_1X$ and $f_2(X) = A_2X$ with

$$A_1 = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -2 & -7 \\ 4 & 4 \end{pmatrix},$$

the pair (f_1, f_2) satisfies the conditions (i), (ii) and (iii) and does not satisfy the condition (iv).

5. RLC Electric Circuit with periodic switching law

In this section, our goal is to illustrate the above theoretical results in a concrete model in order to provide some insight to practitioners in mathematical modeling about how it is possible to apply the theory analyzed in previous section. We will demonstrate that it is not difficult to look for the conditions where it is to be expected the building a center boundary, provided that the Hypotheses HL are fulfilled.

Following the Electrical Example from [1] page 54, after minor changes, we study here a specific model of a RLC Electrical Circuit subordinated to a periodic switching law as shown in the Figure 7 (a). While the switch S is closed, a current i flows in the left-loop and while the switch is open the current vanishes.

Let u be the difference potential between the capacitor plates. By Kirchhoff's rules, the equation from Switched Circuit for the voltage u is

$$LC\frac{d^2u}{dt'^2} + RC\frac{du}{dt'} - (L + M)\frac{di}{dt'} + u = 0.$$

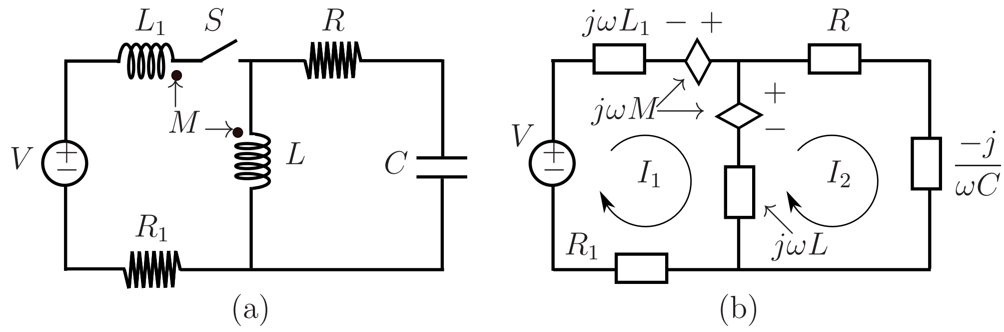

Figure 7: (a) RLC Switched Circuit with a Generator. Being V, L_1, L, R_1, R, C and M respectively: electromotive force, inductance of the left-inductor, inductance of the right-inductor, resistance of the left-resistor, resistance of the right-resistor, capacitance of the capacitor and mutual inductance shared by both inductors. (b) Analogous circuit in the impedances with the switch S closed. Being I_1, I_2 and ω respectively: current in the left-loop, current in the right-loop and frequency of the electromotive force. In order to avoid ambiguity we denote $j = \sqrt{-1}$.

Writing

$$\frac{di}{dt'} = \frac{di}{du}\frac{du}{dt'} = K(u)\frac{du}{dt'},$$

then

$$LC\frac{d^2u}{dt'^2} + (RC - (L+M)K(u))\frac{du}{dt'} + u = 0. \tag{14}$$

If the switch S is open, then $K(u) = 0$. In the case when the switch is closed, let us consider the linear function $i(u) = K_0 u$, with $K_0 \in \mathbb{R}$, where $i(u)$ represents the current i in the dependent variable u. So

$$K(u) = \begin{cases} K_0 & \text{if the switch is closed} \\ 0 & \text{if the switch is open} \end{cases}.$$

A sufficient condition for the function $i(u)$ to be linear, is when the current i and the voltage u are both in the same phase, i.e. $i = i_0\cos(\omega t + \delta)$ and $u = u_0\cos(\omega t + \delta)$. For this given condition, it is enough to consider

$$\omega^2 = \frac{1}{LC}. \tag{15}$$

Effectively, analyzing the equivalent circuit in impedances, Figure 7 (b), we obtain the linear system

$$\left\{ \begin{array}{l} -V + j\omega L_1\hat{I}_1 - j\omega M(\hat{I}_1 - \hat{I}_2) + j\omega M\hat{I}_1 + j\omega L(\hat{I}_1 - \hat{I}_2) + R_1\hat{I}_1 = 0 \\ R\hat{I}_2 - \dfrac{j}{C\omega}\hat{I}_2 + j\omega L(\hat{I}_2 - \hat{I}_1) - j\omega M\hat{I}_1 = 0 \end{array} \right. . \tag{16}$$

Solving (16) in the variables $\hat{I}_1$ and $\hat{I}_2$,

$$\hat{I}_1 = \frac{V\,(RT\omega + j\,(-1 + LC\omega^2))}{\omega\,(L + L_1 + RR_1C - (LL_1 + M^2)\,C\omega^2) + j((L_1R + L(R + R_1))C\omega^2 - R_1)}$$

and

$$\hat{I}_2 = \frac{j(L + M)CV\omega^2}{\omega\,(L + L_1 + RR_1C - (LL_1 + M^2)\,C\omega^2) + j((L_1R + L(R + R_1))C\omega^2 - R_1)}.$$

Consider $Z = \hat{I}_1/\hat{U}$, since $\hat{U} = -(j/(\omega C))\hat{I}_2$ it follows

$$Z = \frac{\omega C\hat{I}_1}{\hat{I}_2} = \frac{RC\omega + j(LC\omega^2 - 1)}{(L - M)\omega} = A + jB.$$

If ω is given by equation (15) then $B = 0$. Therefore the current i and the voltage u are both in the same phase. Furthermore

$$K_0 = \frac{RC}{L - M}.$$

Using the change of variables $(u, \dot{u}, t') \to (-x, y, t)$ then the second order ODE (14) can be written as follows

$$\left\{ \begin{array}{l} \dot{x} = -y \\ \dot{y} = x + \omega((M + L)K - RC)y \end{array} \right. . \tag{17}$$

It is convenient to consider $2h_1 = \omega(RC)$ and $2h_2 = \omega((M + L)K_0 - RC)$, so the two matrices ruling the dynamics in system (17) are

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & -2h_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 2h_2 \end{pmatrix},$$

where he matrices A_1 and A_2 corresponds to the system with the switch closed and open respectively. Note that, if $0 < h_1, h_2 < 1$ then the pair (A_1X, A_2X) satisfies the Hypotheses HL.

Now, we determine a periodic switching law $\sigma : \mathbb{R} \to \{1, 2\}$ for the system (17). Since the pair (A_1X, A_2X) satisfies the Hypotheses HL, by Teorema 2 there are periods (τ_1, τ_2) so that $\gamma_1(-\tau_1, (0, y)) = \gamma_2(\tau_2, (0, y))$. We define time intervals T_n, with $n \in \mathbb{Z}$ in the following way

$$T_0 = 0 \text{ and } T_{n+1} = T_n + \tau_j,$$

where $j = 1$ if n is odd, and $j = 2$ if n is even.

Thus, we define the periodic switching law given by

$$\sigma(t) = \begin{cases} 1 \text{ if } T_{2m+1} \le t \le T_{2m} \\ 2 \text{ if } T_{2m} \le t \le T_{2m+1} \end{cases},$$

where $m \in \mathbb{Z}$. Therefore, if for $t = 0$ the charge on capacitor is empty and decreases on the time t, i.e. $x(0) = 0$ and $y(0) < 0$, then the system $\dot{X} = A_{\sigma(t)}X$ has a continuum of periodic orbits.

Acknowledgments

The first author is partially supported by the grants FP7 PEOPLE-2012-IRSES-318999, Procad-Capes 88881.068462/2014-01 and Fapesp 2013/24541-0. The second author is partially supported by program CAPES/PDSE grant number 7038/2014-03 and CAPES/DS program number 33004153071P0. The third author is supported by program CAPES/PNPD grant number 1271113.

References

[1] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[2] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems − Theory and Applications*, Springer-Verlag (2008).

[3] D.C. Braga, L.F. Mello, *More than three limit cycles in discontinuous piecewise linear differential systems with two zones in the plane*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **24**, 1450056 (2014), 10 pp.

[4] D.C. Braga, L.F. Mello, *Arbitrary number of limit cycles for planar discontinuous piecewise linear differential systems with two zones*, Electronic Journal of Differential Equations, **228** (2015), 1–12.

[5] C.A. Buzzi, C.G. Pessoa and J. Torregrosa, *Piecewise Linear Perturbations of a Linear Center*, Discrete and Continuous Dynamical Systems, **33** (2013), 3915–3936.

[6] S. Coombes, *Neuronal networks with gap junctions: A study of piecewise linear planar neuron models*, SIAM Applied Mathematics, **7** (2008), 1101–1129.

[7] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers-Dordrecht, 1988.

[8] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **8** (1998), 2073–2097.

[9] F. Giannakopoulos and K. Pliete, *Planar systems of piecewise linear differential equations with a line of discontinuity*, Nonlinearity, **14** (2001), 1611–1632.

[10] M. Han and W. Zhang, *On Hopf bifurcation in non-smooth planar systems*, J. of Differential Equations, **248** (2010), 2399–2416.

[11] S.-M. Huan and X.-S. Yang, *The number of limit cycles in general planar piecewise linear systems*, Discrete and Continuous Dynamical Systems, **32** (2012), 2147–2164.

[12] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dynam. Contin. Discrete Impuls. Systems. Ser. B Appl. Algorithms, **19** (2011), 325–335.

[13] J. Llibre, E. Ponce, X. Zhang, *Existence of piecewise linear differential systems with exactly n limit cycles for all $n \in \mathbb{N}$*, Nonlinear Anal. **54** (2003) 977–994.

[14] R. Lum and L. O. Chua, *Global Properties of Continuous Piecewise-Linear Vector Fields. Part I. Simplest Case in $\mathbb{R}^2$*, Memorandum UCB/ERL M90/22, University of California at Berkeley, 1990.

[15] D.D. Novaes and E. Ponce,*A simple solution to the Braga–Mello Conjecture*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **25**, (2015), 155009–7.

[16] C. Pérez and F. Benítez, *Closed curve solutions and limit cycles in a class of second-order switched nonlinear systems*, Systems & Control Letters, **60** (2011), 658–664.

[17] C. Pérez and F. Benítez, *Switched convergence of second-order switched nonlinear systems*, International Journal of Control, Automation and Sistems, **10** (2012), 920–930.

[18] L. Perko, *Differential equations and dynamical systems. Third edition.* Texts in Applied Mathematics **7**. Springer-Verlag, New York, 2001.

[19] X. Xu and P. J. Antsaklis, *Stabilization of second-order LTI switched systems*, International Journal of Control, **72** (2010), 1261–1279.