PIECEWISE LINEAR PERTURBATIONS OF A LINEAR CENTER

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Abstract. This paper is mainly devoted to study the limit cycles that can bifurcate from a linear center using a piecewise linear perturbation in two zones. We consider the case when the two zones are separated by a straight line Σ and the singular point of the unperturbed system is in Σ. It is proved that the maximum number of limit cycles that can appear up to a seventh order perturbation is three. Moreover this upper bound is reached. This result confirm that these systems have more limit cycles than it was expected. Finally, center and isochronicity problems are also studied in systems which include a first order perturbation. For these last systems it is also proved that, when the period function, defined in the period annulus of the center, is not monotone, then it has at most one critical period. Moreover this upper bound is also reached.

1. Introduction and main results

The study of piecewise linear differential systems goes back to Andronov and coworkers [1], and in the recent years a big interest takes place on these systems. Piecewise linear differential systems are used to model many real processes and different modern devices, see for more details [3] and the references therein.

The case of continuous piecewise linear differential systems, when they have only two half-planes separated by a straight line is the simplest possible configuration of piecewise linear differential systems. In 1990 Lum and Chua [23] conjectured that a continuous piecewise linear vector field in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [11].

Limit cycles of discontinuous piecewise linear differential systems defined on two half-planes separated by a straight line have been studied recently in [5, 16, 17, 18, 21, 22], among other papers. Han and Zhang provided discontinuous systems with two limit cycles, and they conjectured that the maximum number of limit cycles for this class is exactly two, see [17]. However, by considering a special family of discontinuous systems sharing the equilibrium position, Huan and Yang in [18] provided strong numerical evidence about the existence of three limit cycles. The example in [18] is the first evidence that linear systems present a configuration with three limit cycles surrounding a unique equilibrium. Later on Llibre and Ponce in [22] provided a proof of the existence of such limit cycles. In the present paper,

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among other results, we present another example obtained simultaneously to the paper by Llibre and Ponce (see [22]), but our approach is completely different. We obtain the limit cycles as a piecewise perturbation of a linear center, and we can choose from which periodic orbits of the linear center the limit cycles bifurcate (see Proposition 3.2). Moreover the limit cycles are hyperbolic.

In the classical theory for smooth systems an important topic is the weak 16th Hilbert problem. The question is: Which is the maximal number of limit cycles that bifurcate from a perturbation of a center? In this field the perturbations of a linear center must be at least quadratic because linear systems do not have limit cycles, but in the class of discontinuous systems we can have limit cycles in piecewise linear systems. In some sense there is a parallelism between the class of smooth quadratic systems and the class of piecewise linear systems in two zones. For example the parameters space has the same dimension, 12. Different canonical forms are presented in [12] for study some bifurcation curves where piecewise linear systems can exibit two limit cycles. In this paper we study the weak 16th Hilbert problem and related problems for this class. More precisely, we consider the following piecewise linear perturbation of a linear center

$$\begin{cases} x' &= ax + by + \sum_{i=1}^{\infty} \varepsilon^i \widetilde{P}_i^{\pm}(x,y), \\ y' &= cx + dy + \sum_{i=1}^{\infty} \varepsilon^i \widetilde{Q}_i^{\pm}(x,y), \end{cases} \tag{1}$$

where $a + d = 0$, $ad - bc > 0$, and the functions $\widetilde{P}_i^{\pm}(x,y)$ and $\widetilde{Q}_i^{\pm}(x,y)$ are polynomials of degree one, defined in the half planes separated by a straight line Σ passing through $(0,0)$.

By a linear change of coordinates and a time rescaling we have that the straight line Σ, in the new coordinates, is given by $\Sigma = \{y = 0\}$ and system (1) becomes

$$\begin{cases} x' &= -y + \sum_{i=1}^{\infty} \varepsilon^i P_i^{\pm}(x,y), \\ y' &= x + \sum_{i=1}^{\infty} \varepsilon^i Q_i^{\pm}(x,y), \end{cases} \tag{2}$$

where $P_i^{\pm}(x,y)$ and $Q_i^{\pm}(x,y)$ are polynomials of degree one, defined in the half planes $\Sigma^+ = \{y \geq 0\}$, $\Sigma^- = \{y \leq 0\}$. We denote the vector fields associated to system (2), defined in $\Sigma^{\pm}$, by $X^{\pm}$, respectively.

This work is an application of a generalization of Françoise's method for smooth systems, see [14]. It is based on a decomposition of certain one-forms associated to the expression of the vector field in polar coordinates. The decomposition, see Section 2, is done in such a way that it simplifies the computations of the first nonzero term, $M_N(\rho)$, of the expansion of the return map associated to the vector field and the positive x-axis so that

$$M(\rho, \varepsilon) = \rho + \varepsilon^N M_N(\rho) + O(\varepsilon^{N+1}). \tag{3}$$

In this case the function $M_N(\rho)$ is called first non-vanishing Poincaré-Pontryagin-Melnikov function. As in smooth systems, for each simple zero of $M_N(\rho)$, ρ^*, there exists a hyperbolic limit cycle γ_ε of the perturbed system (2), such that γ_ε

goes to γ_0 when ε goes to 0 where γ_0 is the level curve $\{x^2 + y^2 = \rho^{*2}\}$ of the unperturbed system. Then the number of zeros of $M_N(\rho)$ determines the upper bound of the number of limit cycles emerging from the center of the unperturbed system up to order N. The method used in this work, see [14], is a generalization to discontinuous systems of the extension for higher order perturbations, see [20], of the method of Françoise, see [8]. The main application in [14] was the computation of the Lyapunov constants for discontinuous systems and their application to the center-focus problem. Other applications of this method can be found in [24, 25]. This procedure is useful not only to discuss the weak 16th Hilbert problem, but also to discuss related problems such that persistence of centers under small perturbations, and study of the period function for centers. In this paper we also get results about these two related problems.

There are some works that show that, for special classes of systems, when the degree in ε of the perturbation increases, then we can obtain more limit cycles. For example, it is well known (see for example [15]) that perturbing the linear center $\dot{x} = -y,\ \dot{y} = x$, by arbitrary polynomials P_n and Q_n of degree n (i.e. by $\dot{x} = -y + \varepsilon P_n(x, y),\ \dot{y} = x + \varepsilon Q_n(x, y)$), we can obtain $[(n-1)/2]$ limit cycles for the perturbed system, where $[\cdot]$ denotes the integer part function. On the other hand, in [19] it is proved that considering $\dot{x} = -y + \sum_{i=1}^{\infty} \varepsilon^i P_i(x, y)$, $\dot{y} = x + \sum_{i=1}^{\infty} \varepsilon^i Q_i(x, y)$ where P_i and Q_i are polynomials of degree n, the maximum number of zeros of $M_N(\rho)$ is $[N(n-1)/2]$. This upper bound, in general, is not reached. In many classes of systems when N increases the number of limit cycles can also increase but, usually, this number stabilizes. The stabilization process depends on the family studied. In [13] this phenomenon was studied for some families of systems, for example, it is presented a concrete class such that when $N = 1, \ldots, 10$, the maximum number of zeros of $M_N(\rho)$ is $0, 0, 1, 1, 1, 2, 2, 2, 2, 2$, respectively. In [19], considering the perturbations of the linear center by quadratic polynomials it is shown that when $N = 1, \ldots, 6$, then the maximum number of zeros of $M_N(\rho)$ is $0, 1, 1, 2, 2, 3$, respectively. A higher order study is not necessary because Bautin in [2], for quadratic systems, proved that at most three limit cycles can appear near a focus or a center. In this paper we observe that this phenomenon also occurs in discontinuous systems. Next result reinforces the parallelism between smooth quadratic systems and piecewise linear systems defined in two zones.

Theorem 1.1. *For system* (2)*, the maximum number of zeros of the corresponding $M_N(\rho)$, is $1, 1, 2, 3, 3, 3, 3$ when $N = 1, \ldots, 7$. Moreover there exist concrete perturbations such that system* (2)*, for ε small enough, exhibit these number of hyperbolic limit cycles up to order N, for $N = 1, \ldots, 7$.*

It is important to mention here that (2) is a small perturbation of a linear center. So the small perturbations keep the infinity as a periodic orbit. In other words, after small perturbations, we do not have singular points at infinity. Thus, if we think the infinity like a point, it can be either a focus or a center. Another related problem to the weak 16th Hilbert problem that we deal in this paper is the persistence of centers under small perturbations. Next result characterizes the centers up to first order perturbations.

Theorem 1.2. *For ε small enough, system*

$$\left\{ \begin{array}{rcl} x' & = & -y + \varepsilon(a_0^{\pm} + a_1^{\pm} x + a_2^{\pm} y), \\ y' & = & x + \varepsilon(b_0^{\pm} + b_1^{\pm} x + b_2^{\pm} y). \end{array} \right. \tag{4}$$

has a center at infinity if and only if there exists a piecewise affine change of variables such that it is reversible with respect to the straight line $y = 0$ or $x = 0$.

We observe that the proof of this result, see Section 4, is constructive and we give explicit conditions involving the coefficients $a_i^{\pm}$ and $b_i^{\pm}$ in order to system (4) to have a center at infinity. Moreover the affine change of variables is also exhibited.

Another related problem is the study of the period function in certain classes of centers. The period function of the center assigns to each periodic orbit its period. If all the periodic orbits have the same period, then the center is called isochronous. The critical periods are the critical points of this function. For smooth systems, Chicone, in [4], conjectured that if a quadratic system has a center with a period function which is not monotone the maximum number of critical periods is two. Partial results for proving this conjecture can be found in [6, 26, 27, 28]. In this paper, for piecewise smooth linear systems, we get the following two results in this direction.

Theorem 1.3. *For ε small enough, consider system (4) with a center at infinity. Then, it is isochronous if and only if, up to a change of variables, X^+ and X^- coincide or both X^+ and X^- have a center or foci at the same point in Σ.*

Theorem 1.4. *For ε small enough, consider system (4) with a center at infinity. Then its period function is either constant, monotone or it has at most one critical period. Moreover, this upper bound is reached.*

This paper is organized as follows. In Section 2 we give the main tools to prove the results of this work. In Section 3 we study the number of limit cycles that appear from piecewise linear perturbations of a linear center. In Section 4 we classify the systems that have centers at infinity that persists up to first order perturbation. In Section 5 we study the period function of the period annulus centers obtained in Section 4. This study includes a classification of the isochronous centers and the explicit conditions on the coefficients of the perturbation that allows a complete description of the period function. All the computations of this work have been done with the Computer Algebra System MAPLE.

2. Preliminary results

This section is devoted to present the main tools that we need to state and prove the results of this paper. Here we follow closely the presentation in [14], which decomposes an arbitrary one-form in polar coordinates. It is reminiscent of the decompositions of [8, 9, 10].

The vector field X given by equation (2), in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, writes as

$$(\dot r, \dot\theta) = \begin{cases} \left(\sum_{i=1}^{\infty} \varepsilon^i R_i^+(r,\theta), 1 + \sum_{i=1}^{\infty} \varepsilon^i \Theta_i^+(r,\theta)\right) & \text{if } \theta \in [0,\pi], \\ \left(\sum_{i=1}^{\infty} \varepsilon^i R_i^-(r,\theta), 1 + \sum_{i=1}^{\infty} \varepsilon^i \Theta_i^-(r,\theta)\right) & \text{if } \theta \in [\pi,2\pi], \end{cases}$$

where $R_i^{\pm}, \Theta_i^{\pm}$ are analytic functions in r, $\sin\theta$ and $\cos\theta$. The solution curves can be also obtained from the system one-forms,

$$\begin{cases} dH + \sum\limits_{i=1}^{\infty} \varepsilon^i \omega_i^+ = 0 & \text{if } \theta \in [0,\pi], \\ dH + \sum\limits_{i=1}^{\infty} \varepsilon^i \omega_i^- = 0 & \text{if } \theta \in [\pi,2\pi], \end{cases} \tag{5}$$

where $H(r) = (x^2 + y^2)/2 = r^2/2$, and $\omega_i^{\pm} = \omega_i^{\pm}(r,\theta)$ are analytic one-forms, 2π–periodic in θ and polynomial in r.

Let $r^+(\theta,\rho)$ (resp. $r^-(\theta,\rho)$) be the solution of X such that $r^+(0,\rho) = \rho$ (resp. $r^-(\pi,\rho) = \rho$). Then, we can define the positive Poincaré half-return map as $\Pi_X^+(\rho) = r^+(\pi,\rho)$, and the negative Poincaré half-return map as $\Pi_X^-(\rho) = r^-(2\pi,\rho)$. The complete Poincaré return map associated to X is given by the composition of these two maps

$$\Pi_X(\rho) = \Pi_X^-(\Pi_X^+(\rho)), \tag{6}$$

see Figure 1. We can also write them in power series of ε as

$$\Pi_X^+(\rho,\varepsilon) = \rho + \sum_{i=1}^{\infty} \varepsilon^i p_i^+(\rho), \text{ and } \Pi_X^-(\rho,\varepsilon) = \rho + \sum_{i=1}^{\infty} \varepsilon^i p_i^-(\rho),$$

and the complete Poincaré return map associated to (2) is given by

$$\Pi_X(\rho,\varepsilon) = \rho + \sum_{i=1}^{\infty} \varepsilon^i p_i(\rho).$$

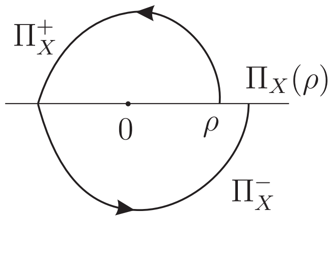

FIGURE 1. Return map of X.

Lemma 2.1. *The first non zero term of the map $\Pi_X(\rho) - \rho$ coincides with the first non-zero term of the map*

$$\Pi_X^+(\rho) - (\Pi_X^-)^{-1}(\rho) = \Pi_X^+(\rho) - \Pi_{\mathcal{R}(X)}^+(\rho),$$

where $\mathcal{R}(X) = (-P(x,-y), Q(x,-y))$ for $X = (P,Q)$. See Figure 2.

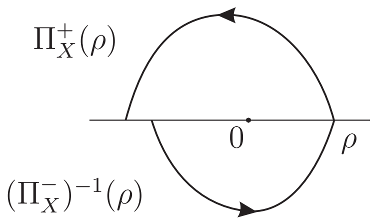

FIGURE 2. Half-return maps Π_X^+ and $(\Pi_X^-)^{-1}$ of (2).

The proof of later lemma follows directly from the reversibility property. That is, for each solution $\gamma(t) = (x(t), y(t))$ of X then $\mathcal{R}(\gamma(t)) = (x(-t), -y(-t))$ is a solution of $\mathcal{R}(X)$.

Then, we only need to study the positive Poincaré half-return map for a smooth planar differential equation,

$$dH + \sum_{i=1}^{\infty} \varepsilon^i \omega_i = 0,$$

but restricted to the region Σ^+.

Lemma 2.2. *Let $\Omega = \alpha(r,\theta)dr + \beta(r,\theta)d\theta$, be an arbitrary analytic one-form, 2π–periodic in θ and $H(r) = r^2/2$. Then there exist functions $h(r,\theta)$, $S(r,\theta)$ and $F(r)$ also 2π–periodic in θ and defined by $F(r) = \frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)d\psi$, $S(r,\theta) = \int_0^\theta \beta(r,\psi)\,d\psi - F(r)\theta$ and $h(r,\theta) = \left(\alpha(r,\theta) - \frac{\partial S(r,\theta)}{\partial r}\right)/H'(r)$, and such that*

$$\Omega = \Omega^0 + \Omega^1 \quad \textit{where } \Omega^0 = hdH + dS,\ \Omega^1 = F(r)d\theta,$$

and

$$\int_{H=\rho^2/2} \Omega^0 = 0, \qquad \int_{H=\rho^2/2} \Omega^1 = \int_{H=\rho^2/2} \Omega.$$

Theorem 2.3. *Let $r(\theta,\varepsilon,\rho)$ be the solution of the initial value problem*

$$\begin{cases} dH + \sum\limits_{i=1}^{\infty} \varepsilon^i \omega_i = 0, \\ r(0,\varepsilon,\rho) = \rho, \end{cases} \tag{7}$$

where $H(r) = r^2/2$ and $\omega_i = \omega_i(r,\theta)$ are one-forms 2π–periodic in θ. Then for any $n \in \mathbb{N}$, $r(\theta,\varepsilon,\rho)$ satisfies the following implicit equation

$$\frac{r^2(\theta,\varepsilon,\rho) - \rho^2}{2} + O(\varepsilon^{n+1}) =$$
$$\sum_{i=1}^{n} \varepsilon^i \left[\int_0^\theta F_i(r(\psi,\varepsilon,\rho))d\psi + S_i(r(\psi,\varepsilon,\rho),\psi)\big|_{\psi=0}^{\psi=\theta} \right],$$

where the one-forms Ω_i and the functions $F_i(r)$, $h_i(r,\theta)$ and $S_i(r,\theta)$ are defined inductively in the following way: $h_0 = 1$,

$$-\Omega_1 := -\omega_1 h_0 = h_1 dH + dS_1 + F_1 d\theta,$$

and

$$-\Omega_i := -\sum_{j=1}^{i} \omega_j h_{i-j} = h_i dH + dS_i + F_i d\theta,$$

for $i = 2, \ldots, n$ and we have used the decomposition given in Lemma 2.2 for the one-forms $-\Omega_i$.

Corollary 2.4. *Let $r(\theta, \varepsilon, \rho) = \sum_{i=0}^{\infty} r_i(\theta, \rho)\varepsilon^i$ the solution of the initial value problem (7). Assume that the functions $r_0(\theta, \rho) = \rho$, $r_1(\theta, \rho)$, $r_2(\theta, \rho), \ldots, r_{n-1}(\theta, \rho)$ are known. Then $r_n(\theta, \rho)$ can be obtained by equating the ε^n–terms of the implicit expression of $r(\theta, \varepsilon, \rho)$ given in Theorem 2.3. In fact the equation looks like*

$$\rho\, r_n(\theta, \rho) = \mathcal{F}_n(\theta, \rho, r_1, \ldots, r_{n-1}),$$

where $\mathcal{F}_n$ depends on the one-forms $\omega_1, \omega_2, \ldots, \omega_n$, through the corresponding F_i, S_i and $r_i = r_i(\theta, \rho)$ $i = 1, 2, \ldots, n$.

In particular $\mathcal{F}_1 = F_1(\rho)\theta + S_1(\rho, \theta) - S_1(\rho, 0)$ and

$$\begin{aligned}\mathcal{F}_2 =& F_2(\rho)\theta + \left(S_2(\rho, \psi) + \frac{\partial S_1}{\partial r}(\rho, \psi) r_1(\psi, \rho) \right)\Bigg|_{\psi=0}^{\psi=\theta} \\ & - \frac{1}{2} r_1(\theta, \rho)^2 + F_1'(\rho) \int_0^{\theta} r_1(\psi, \rho) d\psi.\end{aligned}$$

Proposition 2.5. *The Poincaré-Pontryagin-Melnikov functions of order N is given by*

$$M_N(\rho) = r_N^+(\pi, \rho) - r_N^-(-\pi, \rho),$$

where $r_N^+(\theta, \rho)$ means the coefficient of ε^N of $r(\theta, \varepsilon, \rho)$, given in Corollary 2.4, for the vector field X^+ and $r_N^-(\theta, \rho)$ for the vector field X^-.

The proof of above proposition follows from Lemma 2.1.

We denote by $T^{\pm}(\rho, \varepsilon)$ the flying times on $\Sigma^{\pm}$ of the solution that starts at $\rho \in \Sigma$ then the next result allow us to compute them.

Proposition 2.6. *Under the conditions of Corollary 2.4 we have that*

$$T^{\pm}(\rho, \varepsilon) = \pm \int_0^{\pm\pi} \frac{1}{1 + \sum_{i=1}^{\infty} \varepsilon^i \Theta_i^{\pm}(r(\theta, \varepsilon, \rho), \theta)} d\theta. \tag{8}$$

The proof of this proposition follows from the expression of (2) in polar coordinates. We observe that in (8) both flying times are positive.

3. Number of limit cycles under perturbation

In this section we study the number of limit cycles that appear from piecewise linear perturbations of a linear center. The first result is about the maximum number of zeros of the Poincaré-Pontryagin-Melnikov functions of order N. The proof of Theorem 1.1 follows from Propositions 3.1 and 3.2.

In (3) $M_N(\rho)$ is defined as the first non vanishing term in the power series in ε. Instead of the Poincaré-Pontryagin-Melnikov function of order N only make sense when the previous ones are identically zero, the procedure described in Section 2 allow to obtain expressions for functions $\widehat{M}_k(\rho)$, in terms of the coefficients of the perturbations up to order k, such that $M_1(\rho) = \widehat{M}_1(\rho)$ and

$$M_N(\rho) = \widehat{M}_N(\rho)\Big|_{\{M_k(\rho)\equiv 0, k=1,\ldots,N-1\}}, \text{ for } N \geq 2.$$

With this auxiliary functions we can prove next result.

Proposition 3.1. *For system* (2)*, the maximum number of zeros of the corresponding* $M_N(\rho)$ *is* $1, 1, 2, 3, 3, 3, 3$ *when* $N = 1, \ldots, 7$.

Proof. According to (2), let be $P_i^{\pm}(x, y) = a_{0i}^{\pm} + a_{1i}^{\pm}x + a_{2i}^{\pm}y$ and $Q_i^{\pm}(x, y) = b_{0i}^{\pm} + b_{1i}^{\pm}x + b_{2i}^{\pm}y$. Then

$$M_1(\rho) = \widehat{M}_1(\rho) = \widetilde{M}_1(\rho) = \frac{\pi}{2}(a_{11}^{+} + a_{11}^{-} + b_{21}^{+} + b_{21}^{-})\rho + 2(b_{01}^{+} - b_{01}^{-})$$

and, for $2 \leq k \leq 7$,

$$\widehat{M}_k(\rho) = \frac{\widetilde{M}_k(\rho)}{\rho^{N-2}} = \frac{1}{\rho^{N-2}}\sum_{j=0}^{k-1} A_j^k(\lambda_k)\rho^j, \tag{9}$$

where $A_j^k(\lambda_k)$ is a polynomial of degree k in the variables

$$\lambda_k = (a_{01}^{\pm}, a_{11}^{\pm}, a_{21}^{\pm}, b_{01}^{\pm}, b_{11}^{\pm}, b_{21}^{\pm}, \ldots, a_{0i}^{\pm}, a_{1i}^{\pm}, a_{2i}^{\pm}, b_{0i}^{\pm}, b_{1i}^{\pm}, b_{2i}^{\pm}) \in \mathbb{R}^{12k},$$

for $i = 1, \ldots, k$. We omit the explicit expression of the polynomials $A_j^k(\lambda_k)$ just for simplicity. In Table 1 we show the number of monomials in the variables λ_k for $2 \leq k \leq 7$.

For $N = 1, \ldots, 4$ the polynomials $\widetilde{M}_N(\rho)$ have degree at most $1, 1, 2, 3$, respectively. For $N = 5$, imposing that $\widetilde{M}_k(\rho) \equiv 0$ for $1 \leq k \leq 4$ we obtain that $A_4^5(\lambda_5) \equiv 0$. Then the degree of $\widetilde{M}_5(\rho)$ is at most three. Using the same argument for $N = 6$ and $N = 7$ we obtain that $A_4^6(\lambda_6) \equiv A_5^6(\lambda_6) \equiv A_4^7(\lambda_7) \equiv A_5^7(\lambda_7) \equiv A_6^7(\lambda_7) \equiv 0$. So the degree of the numerators of $\widetilde{M}_6(\rho)$ and $\widetilde{M}_7(\rho)$ are also at most three. The proof concludes because the zeros of M_N and $\widetilde{M}_N$ coincides. ◻

$k\backslash j$	0	1	2	3	4	5	6
2	12	18					
3	4	56	68				
4	8	20	208	204			
5	8	50	94	668	564		
6	12	66	230	324	1916	1422	
7	12	88	330	848	1042	5056	3388

TABLE 1. Number of monomials of the polynomials A_j^k that appear in the proof of Proposition 3.1.

The main obstruction to go further than order seven is the huge expressions that appear in the polynomial system of equations that we have to solve. Table 1 gives an idea of how the complexity increases with N.

Proposition 3.2. *Given $\alpha_1, \alpha_2, \alpha_3 > 0$, for ε small enough, the system*

$$X^+ : \begin{cases} \dot{x} = -y + (1+x)\varepsilon - \dfrac{4\left(\alpha_3\alpha_2 + \alpha_2\alpha_1 + \alpha_3\alpha_1\right)}{3\alpha_3\alpha_2\pi\alpha_1}\varepsilon^2 + \dfrac{2\left(\alpha_1 + \alpha_2 + \alpha_3\right)}{3\alpha_1\alpha_2\alpha_3}\varepsilon^3, \\ \dot{y} = x, \end{cases}$$

$$X^- : \begin{cases} \dot{x} = -y + \varepsilon + \dfrac{16y}{3\alpha_1\alpha_2\alpha_3\pi}\varepsilon^3, \\ \dot{y} = x - y\varepsilon + 2\varepsilon^2 - \dfrac{4\left(\alpha_3\alpha_2 + \alpha_2\alpha_1 + \alpha_3\alpha_1\right)}{3\alpha_3\alpha_2\pi\alpha_1}\varepsilon^3, \end{cases}$$

has three hyperbolic limit cycles. Moreover when ε goes to 0 any limit cycle goes to $x^2 + y^2 = \alpha_i^2$ for each $i = 1, 2, 3$.

Proof. From the algorithm described in Section 2 we get $M_1(\rho) \equiv M_2(\rho) \equiv M_3(\rho) \equiv 0$ and

$$M_4(\rho) = \frac{4(\alpha_1 - \rho)(\alpha_2 - \rho)(\alpha_3 - \rho)}{3\alpha_1\alpha_2\alpha_3\rho^2}.$$

So, α_i for $i = 1, 2, 3$, are simple zeros of $M_4(\rho)$ and the proof is completed. □

From the computations of the proof of Proposition 3.1 we can choose concrete values for the coefficients of the polynomials of the perturbations in order to obtain explicit systems, as the system of Proposition 3.2, such that the maximum number of limit cycles that bifurcate from system (2) up to order N for $N = 1, 2, 3$ is $1, 1, 2$, respectively.

4. Centers of first order perturbation

It is well known that a smooth linear system with an isolated monodromic singular point has not singular points at infinity, that is the infinity is a periodic orbit in the Poincaré compactification, see [7]. Hence, in analogy with the smooth case, we have that the infinity of system (1), or in its simplified form (4), is a periodic orbit for ε small enough. Thus, if we think the infinity like a point, it can be either a focus or a center. For ε small enough we will see that, for system (4), there exists $\rho_0 > 0$ such that the return map (6) always is defined for all ρ with $\rho > \rho_0$. Therefore the infinity is a center of system (1) if and only if the return map (6) is the identity map.

In this section we classify the centers at infinity of discontinuous first order perturbation of linear center, i.e., we give necessary and sufficient conditions to determine when the return map of the system (4) is the identity map. Therefore, using the method of Section 2, we need to determine when all terms of order bigger than one of the return map vanish.

Before to prove the main result of this section, we prove the following lemma that gives an explicit expression of the half-return map for a linear system.

Lemma 4.1. *Consider the linear vector field Y associated to system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{10}$$

with initial conditions $x(0) = \rho$ and $y(0) = 0$. Suppose that the trace vanishes and the determinant is positive ($d = -a$ and $-a^2 - bc > 0$). Then the positive Poincaré half-return map is

$$\Pi_Y^+(\rho) = \begin{cases} -\rho - \dfrac{2(b\beta + a\alpha)}{a^2 + bc} & \text{if } a\beta - c\alpha = 0, \\ -\rho - \dfrac{2\beta}{c} & \text{if } a\beta - c\alpha \neq 0. \end{cases}$$

Moreover its negative Poincaré half-return map $(\Pi_Y^-(\rho))^{-1}$ have the same expression.

Proof. We have that the solution of the initial value problem (10) is given by

$$\begin{aligned} x(t) &= \frac{A\nu - C}{\nu}\cos(\nu t) + \frac{B}{\nu}\sin(\nu t) + \frac{C}{\nu}, \\ y(t) &= \frac{-\widetilde{C}}{\nu}\cos(\nu t) + \frac{\widetilde{B}}{\nu}\sin(\nu t) + \frac{\widetilde{C}}{\nu}, \end{aligned} \tag{11}$$

where $A = \rho$, $B = a\rho + \alpha$, $C = b\beta + a\alpha$, $\widetilde{B} = c\rho + \beta$, $\widetilde{C} = c\alpha - a\beta$ and $\nu = -a^2 - bc > 0$.

If $a\beta - c\alpha = 0$, then $\widetilde{C} = 0$ and $t = \frac{\pi}{\nu}$ is the smallest positive solution of $y(t) = 0$ in (11). Hence, evaluating $t = \frac{\pi}{\nu}$ in $x(t)$, in (11), we obtain the expression of the positive Poincaré half-return map $\Pi_Y^+(\rho)$ given in the statement.

When $a\beta - c\alpha \neq 0$, system (11), with the change $(\cos(\nu t), \sin(\nu t)) = (\kappa, \sigma)$, writes as

$$\begin{aligned} (A\nu - C)\kappa + B\sigma &= x\nu - C, \\ -\widetilde{C}\kappa + \widetilde{B}\sigma &= -\widetilde{C}, \\ \kappa^2 + \sigma^2 &= 1. \end{aligned}$$

The determinant of the matrix of the linear system in variables (κ, σ) formed by the two first equations never vanishes, because it is a quadratic polynomial on ρ with negative discriminant, $4(a\beta - c\alpha)^2(a^2 + bc) < 0$. Therefore it has the unique solution

$$\kappa = \frac{\widetilde{B}x\nu + B\widetilde{C} - \widetilde{B}C}{A\widetilde{B}\nu + B\widetilde{C} - \widetilde{B}C}, \; \sigma = \frac{-\widetilde{C}\sqrt{\nu}(-x + A)}{A\widetilde{B}\nu + B\widetilde{C} - \widetilde{B}C},$$

satisfying $\kappa^2 + \sigma^2 = 1$. The solutions of this last equation are

$$x = \rho, \quad \text{and} \quad x = -\rho - \frac{2\beta}{c}.$$

The first one corresponds to $t = 0$ and the second one give us the positive Poincaré half-return map.

In the above computations if we change t by $-t$ in (11), we obtain the same result for the negative Poincaré half-return map. ∎

For smooth systems an affine change of variables, including a time rescaling, reduces the number of parameters of the vector field. In piecewise differential systems we can make this kind of changes, include different times rescaling, on each zone but, in order to preserve the qualitative behavior of the global phase portraits, we need to impose some restrictions. We detail them only in the two zones case.

(i) The matrix of any affine change must have non vanishing determinant.
(ii) Each point of Σ must be transformed for both affine changes into the same point in Σ.
(iii) All the real and virtual singular points of the system must remain on the same side as they were.
(iv) The rescaling times must have the same sign on both zones.

When $\Sigma = \{y = 0\}$, all these conditions can be achieved by means of a *global change of variables* $(x, y, t) \mapsto (u, v, \tau)$ defined in $\Sigma^{\pm}$ by the piecewise function

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B^{\pm} \\ 0 & C^{\pm} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D \\ 0 \end{pmatrix} \tag{12}$$

and $\tau = E^{\pm} t$, where $E^+E^- > 0$ (condition (iv)), $(AC^+)(AC^-) \neq 0$ (condition (i)) and $C^+ > 0$, $C^- > 0$ (condition (iii)). Moreover, in order to preserve the time, we restrict this change to $E^{\pm} = 1$. This change simplify the expression of the vector fields that appear in the proof of the next results.

Theorem 4.2. *For ε small enough, the following statements are equivalent:*
(a) System (4) *has a center at infinity.*
(b) System (4) *satisfies one of following conditions:*
  (i) $b_0^- = b_0^+ = a_1^- + b_2^- = a_1^+ + b_2^+ = 0$.
  (ii) $b_0^- - b_0^+ = b_1^- - b_1^+ = a_1^- + b_2^- = a_1^+ + b_2^+ = 0$ *and* $b_0^+ \neq 0$.
  (iii) $b_0^- = b_0^+ = a_0^- + a_0^+ = a_1^- + a_1^+ = b_1^+ + a_2^- = b_1^- + a_2^+ = b_2^- + b_2^+ = 0$ *and* $a_1^+ + b_2^+ \neq 0$.
  (iv) $b_0^- = b_0^+ = a_0^- + a_0^+ = (a_1^+ + b_2^-)(a_1^- + a_1^+) - (b_1^- + a_2^+)(a_2^- - a_2^+) = (a_1^- + b_2^+)(a_1^- + a_1^+) + (b_1^- + a_2^+)(a_2^- - a_2^+) = -b_1^- + b_1^+ + a_2^- - a_2^+ = 0$ *and* $(a_1^- + a_1^+)(a_1^+ + b_2^+) \neq 0$.
  (v) $a_0^- + a_0^+ = b_0^- - b_0^+ = a_1^- + a_1^+ = b_1^- - b_1^+ = a_2^- - a_2^+ = b_2^- + b_2^+ = 0$ *and* $a_1^+ + b_2^+ \neq 0$.
(c) Up to a change of variables of type (12), *system* (4) *is reversible with respect to the straight lines $y = 0$ or $x = 0$.*

Proof. Assume that system (4) has a center at infinity, then the functions $M_N(\rho)$ given in (3) vanish identically for all N. From the algorithm described in Section 2 we can compute all the expression for the functions $M_N(\rho)$.

The first condition is

$$M_1(\rho) = \frac{(b_2^+ + b_2^- + a_1^- + a_1^+)\pi\rho}{2} + 2(b_0^+ - b_0^-) \equiv 0.$$

Then $b_0^- = b_0^+$ and $b_2^- = -b_2^+ - a_1^- - a_1^+$.

Now we add the second condition

$$\begin{aligned} M_2(\rho) = & \frac{-(a_1^+ + b_2^+)(b_1^+ - a_2^+ - b_1^- + a_2^-)\pi\rho}{4} \\ & + 2(b_0^+(b_1^- - b_1^+) + (a_0^- + a_0^+)(a_1^+ + b_2^+)) \equiv 0. \end{aligned}$$

If $a_1^+ + b_2^+ = 0$ then either $b_0^+ = 0$ or $b_0^+ \neq 0$ and $b_1^+ = b_1^-$. That it corresponds with the cases (i) and (ii), respectively.

All the other cases satisfy $a_1^+ + b_2^+ \neq 0$. With this restriction we write

$$b_1^+ = a_2^+ + b_1^- - a_2^- \text{ and } a_0^+ = \frac{b_0^+(a_2^+ - a_2^-) - a_0^-(a_1^+ + b_2^+)}{a_1^+ + b_2^+},$$

and the third condition writes as

$$\begin{aligned} M_3(\rho) = &- \frac{(a_1^+ + b_2^+)\left((a_2^- - a_2^+)(b_1^- + a_2^+) + (a_1^- + a_1^+)(a_1^- + b_2^+)\right)\pi\rho}{4} \\ &- 2b_0^+\left((a_2^- - a_2^+)(b_1^- + a_2^+) + (a_1^- + a_1^+)(a_1^+ + b_2^+)\right) \\ &+ \frac{b_0^+(a_2^- - a_2^+)\left(b_0^+(a_2^- - a_2^+) + 2a_0^-(a_1^+ + b_2^+)\right)\pi}{2(a_1^+ + b_2^+)\rho} \equiv 0. \end{aligned}$$

First we consider the case $b_0^+ = 0$, consequently $a_0^+ = -a_0^-$, and $M_3(\rho) \equiv 0$ reduces to condition

$$(a_2^- - a_2^+)(b_1^- + a_2^+) + (a_1^- + a_1^+)(a_1^- + b_2^+) = 0. \tag{13}$$

Now there are three cases to consider: $a_1^- + a_1^+ = b_1^- + a_2^+ = 0$ that corresponds to case (iii), $a_1^- + a_1^+ = a_2^- - a_2^+ - = 0$ that it is included in the case (v) and case (iv) that corresponds to $a_1^- + a_1^+ \neq 0$ and (13).

Secondly we study the last case $b_0^+ \neq 0$. When $a_2^- - a_2^+ = 0$, vanishing the other coefficients of $M_3(\rho)$, we obtain the last class (v), which corresponds to the condition $a_1^- + a_1^+ = 0$. Straightforward computations show that the other cases correspond to case (v) with additional assumptions, $a_0^+ = a_0^- = 0$.

Statement (c) follows directly from (b) applying a change of variables of type (12) and observing that the first two cases are reversible with respect to $x = 0$ and the remaining ones are reversible with respect to $y = 0$. The changes are
$\{A = C^+ = C^- = 1, B^\pm = -a_1^\pm\varepsilon/(b_1^-\varepsilon + 1), D = b_0^-\varepsilon/(b_1^-\varepsilon + 1),\}$,
$\{A = C^+ = C^- = 1, B^\pm = -a_1^\pm\varepsilon/(b_1^\pm\varepsilon + 1), D = 0\}$,
$\{A = C^+ = 1, B^+ = B^- = D = 0, C^- = (1 + \varepsilon b_1^+)/(1 + \varepsilon b_1^-)\}$,
$\{A = C^+ = 1, B^+ = D = 0, B^- = \varepsilon(a_1^- + a_1^+)/(1 + \varepsilon b_1^-), C^- = (1 + \varepsilon b_1^+)/(1 + \varepsilon b_1^-)\}$
and the identity, respectively.

The proof finishes using the reversibility property because for ε small enough system (4) remains monodromic. □

Theorem 1.2 is an immediate consequence of the previous result.

Remark 4.3. *Lemma 4.1 gives an alternative proof for the cases (i) and (ii).*

As system (4) is a perturbation of a liner center, the singular point of $X^+(X^-)$ is either a center (C) or a focus (F). Then we will say that the perturbed system (4) is of CC, FF or FC-type when both are centers, both are foci or one is a focus and the other one is a center.

Corollary 4.4. *Consider system* (4)*, for ε small enough, then the following statements hold:*

(a) *Families (i-ii) of Theorem 4.2 are of CC-type and the phase portraits are topologically equivalent to one of the pictures* (a), (b) *or* (c) *of Figure 3;*
(b) *Families (iii-v) of Theorem 4.2 are of FF-type and the phase portraits are topologically equivalent to one of the pictures* (b), *or* (d) *of Figure 3;*

Proof. Firstly, we observe that for any of the families of Theorem 4.2 the trajectories of the vector fields $X^{\pm}$ have a unique tangency, with Σ, that coincides. The trace of $X^{\pm}$ vanishes for families (i) and (ii). Therefore, the phase portraits are linear centers in both zones. That is system (4) is of CC-type. For the remaining cases the trace of $X^{\pm}$ does not vanish but the sum of the traces is zero. So, system (4) it of FF-type and the phase portraits are symmetric foci with respect to Σ. □

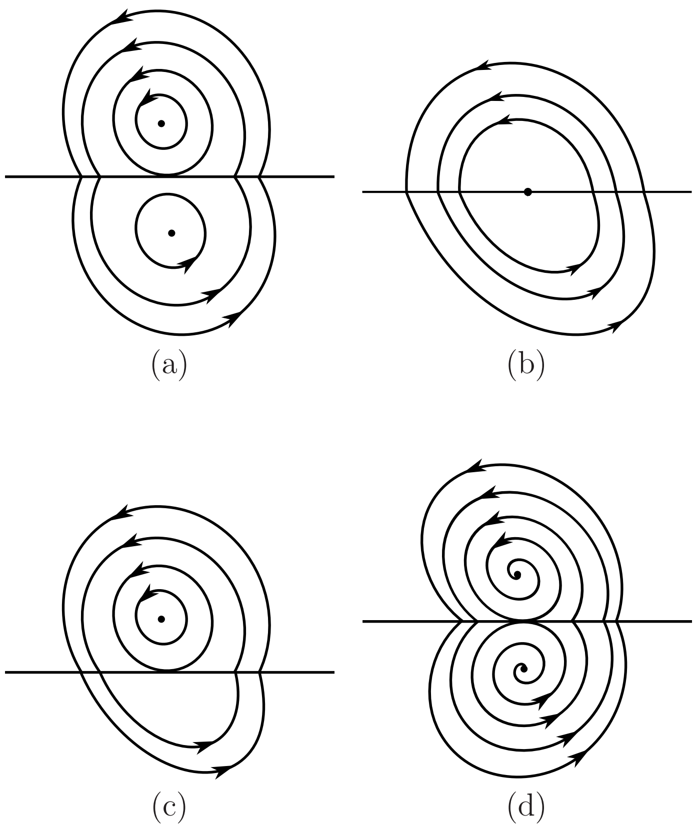

FIGURE 3. Phase portraits.

5. ISOCHRONOUS CENTERS AND OSCILLATIONS OF THE PERIOD FUNCTION

As we have defined in Section 2, when the complete Poincaré return map exists, we can associate to any orbit which start at $(0, \rho)$, in polar coordinates, the corresponding flying times $T^{\pm}$ on $\Sigma^{\pm}$ and the period, in power series of ε, writes as

$$T(\rho, \varepsilon) = T^{+}(\rho, \varepsilon) + T^{-}(\rho, \varepsilon) = \sum_{i=0}^{\infty} T_i(\rho)\varepsilon^i. \tag{14}$$

We recall that an isochronous center is a center such that all of its periodic orbits have the same period. Then, an equivalent condition, is that the derivative of $T(\rho, \varepsilon)$ with respect to ρ is identically zero. For each family in Theorem 4.2 we find conditions on the coefficients of the vector field in order to characterize the isochronous centers at infinity, that is all the centers at infinity such that $T'_i(\rho) \equiv 0$ for all i.

Theorem 5.1. *For ε small enough, the following statements are equivalent:*

(a) System (4) *has an isochronous center at infinity.*

(b) System (4) *satisfies one of the following conditions:*

- *(i)* $a_0^- = a_0^+ = b_0^- = b_0^+ = a_1^- + b_2^- = a_1^+ + b_2^+ = 0$.
- *(ii)* $b_0^- = b_0^+ = a_0^- - a_0^+ = a_1^- + b_2^- = a_1^+ + b_2^+ = b_1^- - b_1^+ - a_2^- + a_2^+ = (b_1^- - b_1^+)(b_1^- + a_2^+) + (a_1^- - a_1^+)(a_1^- + a_1^+) = 0$ *and* $a_0^+ \neq 0$.
- *(iii)* $a_0^- - a_0^+ = b_0^- - b_0^+ = b_1^- - b_1^+ = a_1^- + b_2^- = a_1^+ + b_2^+ = a_2^- - a_2^+ = b_2^- - b_2^+ = 0$ *and* $b_0^+ \neq 0$.
- *(iv)* $a_0^- = a_0^+ = a_1^- = a_1^+ = b_2^- = b_2^+ = b_0^- - b_0^+ = b_1^- - b_1^+ = 0$ *and* $b_0^+ \neq 0$.
- *(v)* $a_0^- = a_0^+ = b_0^- = b_0^+ = a_1^- + a_1^+ = b_1^+ + a_2^- = b_1^- + a_2^+ = b_2^- + b_2^+ = 0$ *and* $a_1^+ + b_2^+ \neq 0$.
- *(vi)* $a_0^- = a_0^+ = b_0^- = b_0^+ = (a_1^+ + b_2^-)(a_1^- + a_1^+) - (b_1^- + a_2^+)(a_2^- - a_2^+) = (a_1^- + b_2^+)(a_1^- + a_1^+) + (b_1^- + a_2^+)(a_2^- - a_2^+) = -b_1^- + b_1^+ + a_2^- - a_2^+ = 0$ *and* $(a_1^- + a_1^+)(a_1^+ + b_2^+) \neq 0$.
- *(vii)* $a_0^- = a_0^+ = b_0^- = b_0^+ = a_1^- + a_1^+ = b_1^- - b_1^+ = a_2^- - a_2^+ = b_2^- + b_2^+ = 0$ *and* $a_1^+ + b_2^+ \neq 0$.
- *(viii)* $a_0^- = a_0^+ = a_1^- = a_1^+ = b_0^- - b_0^+ = b_1^- - b_1^+ = a_2^- - a_2^+ = b_2^- + b_2^+ = 0$ *and* $b_2^+ b_0^+ \neq 0$.

(c) System (4) *have a center at infinity and either, up to a change of variables* (12)*, X^+ and X^- coincide or both X^+ and X^- have a center or foci at the same point in Σ.*

Moreover, the phase portraits are given by pictures (b) *and* (c) *of Figure 3.*

Proof. Assume that system (4) has a center at infinity, then from the algorithm described in Section 2 and Proposition 2.6 we compute all the necessary functions $T_i(\rho)$ given in (14). We only show, for simplicity, the first two: $T_0(\rho) = 2\pi$ and

$$T_1(\rho) = \frac{4(a_0^+ - a_0^-)}{\rho} + \frac{1}{2}(a_2^+ + a_2^- - b_1^+ - b_1^-)\pi\rho.$$

Assume that we are in the family (ii) of Theorem 4.2. Computing the derivative $T'_1(\rho)$ we obtain $2(a_0^- - a_0^+)$. So, we add the condition $a_0^+ = a_0^-$. Now $T'_2(\rho)$, $T'_3(\rho)$ and $T'_4(\rho)$ vanish identically when

$$a_0^-(a_2^- - a_2^+) + b_0^+(b_2^- - b_2^+) = 0, \tag{15}$$

$$b_0^+ b_2^+(a_2^- - a_2^+) + a_0^- \left((b_1^+ + a_2^+)(a_2^- - a_2^+) + (b_2^- - b_2^+)(b_2^- + b_2^+)\right) = 0 \tag{16}$$

and

$$a_0^- b_2^- b_2^+(a_2^- - a_2^+) = (a_0^-)^3(a_2^- - a_2^+) = 0,$$

respectively. We have two cases to consider: either $a_2^- - a_2^+ = 0$ or $a_2^- - a_2^+ \neq 0$ and $a_0^- = 0$. In the first case the conditions (15) and (16) become

$$b_0^+(b_2^- - b_2^+) = 0 \text{ and } a_0^-(b_2^- - b_2^+)(b_2^- + b_2^+) = 0.$$

So $b_2^- = b_2^+$ because $b_0^+ \neq 0$. This is the case (iii) of the statement (b). In the second case the conditions (15) and (16) become

$$b_0^+(b_2^- - b_2^+) = 0 \text{ and } b_0^+ b_2^+(a_2^- - a_2^+) = 0.$$

So, as $b_0^+ \neq 0$ and $a_2^- - a_2^+ \neq 0$, we get $b_2^- = b_2^+ = 0$. These conditions correspond to case (iv) of the statement (b).

The other families of Theorem 4.2 follow similarly computing $T_i'(\rho)$ up to $i = 3$, 1, 1 and 2 for the families (i), (iii), (iv) and (v), respectively.

Assume that system (4) satisfies one of the conditions stated in (b). For families (i) and (v-viii) it is clear that the equilibrium point of X^+ and X^- coincide at the same point in Σ. Family (iv) satisfies $X^+ = X^-$ and for families (ii-iii) straightforward computations show that there exist changes of variables of type (12) such that, in the new coordinates, $X^+ = X^-$. Then statement (c) is proved.

Finally, assume that system (4) satisfy the conditions stated in (c). When the equilibrium point of X^+ (X^-) remains in Σ the flying time T^+ (T^-) coincides for all orbits in X^+ (X^-). Then the function $T(\rho, \varepsilon)$ is constant for all ρ. This last property is also satisfied when, up to a change of variables (12), $X^+ = X^-$ because this change preserves the time of the equation. Then statement (a) follows because system (4) has a center at infinity. □

The proof of Theorem 1.3 follows immediately from later theorem. Next result deals with the number of critical periods for the centers given in Theorem 4.2 and it implies also Theorem 1.4.

Theorem 5.2. *Assume that, for ε small enough, system* (4) *has a center at infinity. Then the following statements hold.*

(i) If it is of FF-type then the singular points can be either both real, or both virtual or both coincide in Σ and the period function is decreasing, increasing and constant, respectively.

(ii) If it is of CC-type, when the singular points
   - *(a) are both real or one real and the other in Σ, then the period function is decreasing,*
   - *(b) are both virtual or one virtual and the other in Σ, then the period function is increasing,*
   - *(c) both coincide in Σ, then the period function is constant,*
   - *(d) are one real and the other one virtual, then it has at most one critical period. Moreover, this upper bound is reached.*

Proof. Let $p^\pm$ be the singular points of the vector fields $X^\pm$ and $\alpha^\pm \pm \beta^\pm i$ are the complex eigenvalues of the matrices associated to $X^\pm$. Let $(\rho, 0)$ be a point in Σ such that $\rho_0 < \rho$ where the point $(\rho_0, 0)$ is the unique point that the vector field X^+ is tangent to Σ. Then we denote by $\overline{\rho} = \Pi_X^+(\rho)$ and by $\Psi^+(\rho)$ the external (internal) angle on the vertex p^+ of the triangle formed by the points $(\rho, 0)$, p^+

and $(\overline{\rho},0)$ when p^+ is real (virtual). When p^+ is in Σ then we define $\Psi^+(\rho)=\pi$. It is clear that the flying time in Σ^+ is $T^+(\rho)=\Psi^+(\rho)/\beta^+$. Note that when p^+ is real $\Pi_X^+(\rho_0)\neq\rho_0$ but when p^+ is virtual or is in Σ then $\Pi_X^+(\rho_0)=\rho_0$. Then, given a point $(\rho,0)$ such that $\rho<\Pi_X^+(\rho_0)$, in an analogous way we can define $\Psi^-(\rho)$ and, consequently, $T^-(\rho)=\Psi^-(\rho)/\beta^-$.

Next properties can be checked easily. See them at Figure 4.

(i) If p^+ is a real singular point then Ψ^+ is a decreasing function such that $\Psi^+(\rho)\in[\pi,2\pi]$ for all $\rho_0<\rho$ and $T^+(\rho)$ goes to π/β^+ when ρ goes to ∞. Then T^+ is a decreasing function.
(ii) If p^+ is on Σ then $\Psi^+(\rho)=\pi$ for all $\rho_0<\rho$ and, consequently, T^+ is a constant function.
(iii) If p^+ is a virtual singular point then Ψ^+ is an increasing function such that $\Psi^+(\rho)\in[0,\pi]$ for all $\rho_0<\rho$ and $T^+(\rho)$ goes to π/β^+ when ρ goes to ∞. Then T^+ is an increasing function.

All the statements except (ii.d) follows immediately from the fact that $T(\rho)=T^+(\rho)+T^-(\rho)$ and applying later properties (i-iii) to the families given in Corollary 4.4.

The period function for the remaining case, (ii.d), is the unique which can present oscillations because one of the singular points is real and the other one is virtual. From Theorem 4.2 there exists a time-preserving change of variables such that system (4) is reversible with respect to $x=0$. Then it is not restrictive to assume that p^+ is real and p^- is virtual. The reversibility ensures that the triangles formed by $(\rho,0)$, $p^\pm$ and $(\overline{\rho},0)=(-\rho,0)$ are isosceles. Then, if $\delta^\pm$ are the distances from $p^\pm$ to Σ, the period function is given by

$$T(\rho)=T^+(\rho)+T^-(\rho)=\frac{2\pi}{\beta^+}-\frac{2}{\beta^+}\arctan\left(\frac{\rho}{\delta^+}\right)+\frac{2}{\beta^-}\arctan\left(\frac{\rho}{\delta^-}\right),$$

where arctan is the usual inverse of the tangent function, and its derivative is given by

$$T'(\rho)=-\frac{2}{\beta^+\left((\delta^+)^2+\rho^2\right)}+\frac{2}{\beta^-\left((\delta^-)^2+\rho^2\right)},$$

and there exists at most one value $\rho^*>0$ such that $T'(\rho^*)=0$, then system (4) has at most one critical period.

The proof ends checking that the period function associated to system

$$X^+:\begin{cases}\dot{x}=-y+\varepsilon,\\ \dot{y}=x,\end{cases}\quad\text{and}\quad X^-:\begin{cases}\dot{x}=-y+4\varepsilon,\\ \dot{y}=x\end{cases}\tag{17}$$

has exactly one critical period at $\rho^*=2\varepsilon$ because $\delta^+=\varepsilon$, $\delta^-=4\varepsilon$, $\beta^+=\beta^-=1$ and $T'(\rho)=\dfrac{6\varepsilon(\rho^2-4\varepsilon^2)}{\rho^4+17\varepsilon^2\rho^2+16\varepsilon^4}$. □

As a final remark we observe that the period function associated to system (17) has a unique minimum because $T''(\rho^*)=6/(25\varepsilon^2)>0$. Examples with only one maximum can be also obtained.

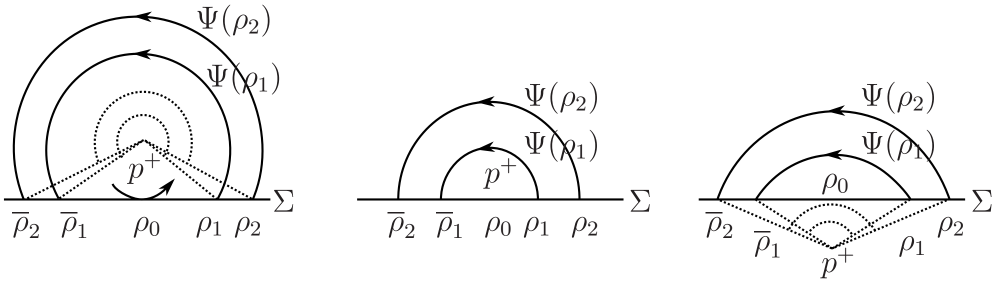

FIGURE 4. The angle of half-return map in terms of start and end points

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