NEWTON'S METHOD FOR SYMMETRIC QUARTIC POLYNOMIALS

BEATRIZ CAMPOS, ANTONIO GARIJO, XAVIER JARQUE, AND PURA VINDEL

Abstract. We investigate the parameter plane of the Newton's method applied to the family of quartic polynomials $p_{a,b}(z) = z^4 + az^3 + bz^2 + az + 1$, where a and b are real parameters. We divide the parameter plane $(a, b) \in \mathbb{R}^2$ into twelve open and connected *regions* where p, p' and p'' have simple roots. In each of these regions we focus on the study of the Newton's operator acting on the Riemann sphere.

Keywords: Newton's method, holomorphic dynamics, Julia and Fatou sets.

1. Introduction

Newton's method is the universal root finding algorithm in all scientific areas of knowledge. It is also the seed of what we know as holomorphic dynamics and it goes back to Ernest Schröder and Artur Caley who investigated the global dynamics of Newton's method applied to low degree polynomials as a rational map defined on the Riemann sphere. This global study is not only theoretical but it also has important implications at computational level (see for instance [HSS01]).

Given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, we consider the dynamical system given by the iterates of f. The Riemann sphere splits into two totally $f-$invariant subsets: the *Fatou set* $\mathcal{F}(f)$, which is defined to be the set of points $z \in \hat{\mathbb{C}}$ where the family $\{f^n, \quad n \geq 1\}$ is normal in some neighborhood of z, and its complement, the *Julia set* $\mathcal{J}(f) = \hat{\mathbb{C}} \setminus \mathcal{F}(f)$. The Fatou set is open and therefore $\mathcal{J}(f)$ is closed. Moreover, if the degree of the rational map f is greater than or equal to 2, then the Julia set $\mathcal{J}(f)$ is not empty and it is the closure of the set of repelling periodic points of f.

The connected components of $\mathcal{F}(f)$, called *Fatou components*, are mapped under f among themselves. If follows from the Classification Theorem ([Mil06], Theorem 13.1) that any periodic Fatou component of a rational map is either the basin of attraction of an attracting or parabolic cycle or a simply connected rotation domain (a Siegel disk) or a doubly connected component rotation domain (a Herman ring). Moreover, the basin of attraction of an attracting or parabolic cycle contains, at least, one critical point i.e. a point $z \in \hat{\mathbb{C}}$ such that $f'(z) = 0$. For a background on the dynamics of rational maps we refer to [Mil06, Bea91, CG93].

Given p a polynomial of degree $d \geq 2$ we define the Newton's map as

$$N_p(z) := z - \frac{p(z)}{p'(z)}.$$

Clearly, roots of p correspond to attracting fixed points of N_p. It is well-known (see [Shi09]) that $\mathcal{J}(N_p)$ is connected (see also [BFJK14, BFJK]) and consequently, all Fatou components are simply connected. Although, as we claimed, Newton's method is *the* universal root finding algorithm it reveals limitations. Precisely, for some polynomials of degree $d \geq 3$, there are open

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sets of initial conditions in the dynamical plane not converging to any root of p. The reason for this is the existence of at least one *free* critical point which allows the Newton's map of p (for an open set of polynomials in parameter space) to have attracting basins of period $k \geq 2$. Of course for all seeds on those basins the iterates are not converging to any of the roots of p. See [McM87] for a remarkable discussion in this direction.

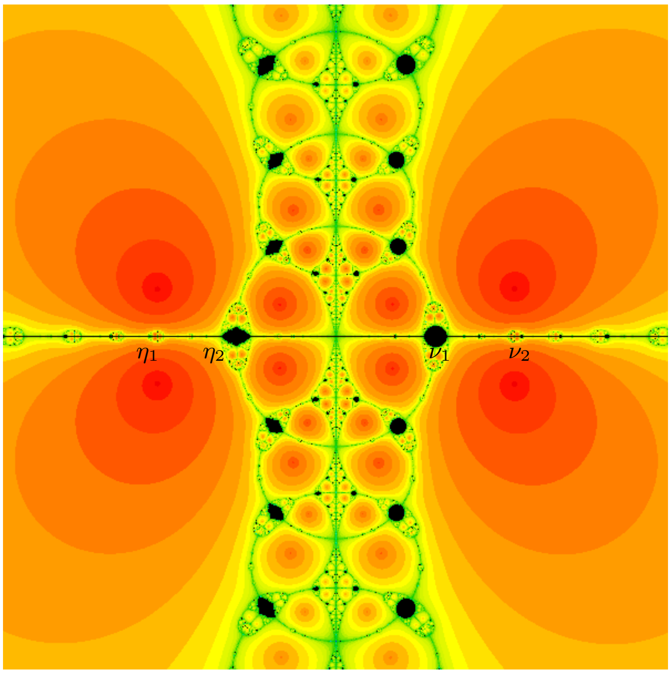

FIGURE 1. Dynamical plane of $N_{a,b}$ for $a = 0.0013$ and $b = -1.73729$ in the region $\mathcal{R}_4$. We can see in red the four basins of attractions of the complex roots of $p_{a,b}$ and in black the two attracting cycles of period two, $\{\eta_1, \eta_2\}$ and $\{\nu_1, \nu_2\}$, to which the *free* critical points are attracted.

There are also several results about the dynamical plane as well as the parameter plane of Newton's method applied to some concrete families of polynomials. The most studied case is Newton's method of cubic polynomials $q(z) = z(z-1)(z-a)$, $a \in \mathbb{C}$, for which the only *free* critical point is located at $(a+1)/3$. See [Tan97, Roe08] and references therein.

However, there is not a general study on Newton's method for quartic polynomials. Since the degree of N_p in this case is $d = 4$ (we assume no double roots of p to keep inside the family), we know there are $2d - 2 = 6$ critical points. Of course four of them correspond to the four roots of p but we have two *free* extra critical points and hence the parameter space is $\mathbb{C}^2$. In our approach we decrease the dimension of the parameter space but somehow we keep the difficulty. Indeed, we consider the family of symmetric quartic polynomials with two real parameters. Thus, although the parameter space is $\mathbb{R}^2$, the number of *free* critical points is still two. Figure 1 illustrates one case where the two critical points are attracted by two different period two attracting cycles.

Precisely, the main goal of this paper is to study some topological properties of the parameter and dynamical plane of Newton's method applied to the family of four degree symmetric polynomials:

$$p_{a,b} := p_{a,b}(z) = z^4 + az^3 + bz^2 + az + 1, \tag{1}$$

when a and b are real parameters. Symmetric polynomials frequently appear along the dynamical study of others families of iterative methods (see [CCTV15], for example). We will split the parameter plane into regions in which the roots of the polynomials p, p' and p'' have simple zeroes and determine in which of those parameter regions we can guarantee that, except for a measure zero set (which is no relevant from the numerical point of view), any seed in dynamical plane converges to a root of p (Proposition 1 and Proposition 2). We also give numerical evidences that in other regions a more complicated and chaotic dynamics is possible. From the theoretical point of view these results are a first step in the direction of having a better understanding of Newton's method applied to quartic polynomials.

The expression of the Newton's map applied to $p_{a,b}$ writes as

$$N := N_{a,b}(z) = z - \frac{p_{a,b}(z)}{p'_{a,b}(z)} = z - \frac{z^4 + az^3 + bz^2 + az + 1}{4z^3 + 3az^2 + 2bz + a}. \tag{2}$$

The critical points of N are the solutions of $N'(z) = 0$; that is, the roots of p and p''. For each root $r_i(a, b) := r_i$, $i = 1, \cdots, 4$ we define its *basin of attraction*, $\mathcal{A}_{a,b}(r_i)$, as the set of points in the complex plane which tend to r_i under the Newton's map iteration. That is

$$A_{a,b}(r_i) = \{z \in \mathbb{C}\,,\; N^k_{a,b}(z) \to r_i \text{ as } k \to \infty\}.$$

In general, $A_{a,b}(r_i)$ may have infinitely many connected components but only one of them, denoted by $A^*_{a,b}(r_i)$ and called *immediate basin of attraction of* r_i, contains $z = r_i$. See Figure 1.

The paper is structured as follows. In Section 2 we investigate in the parameter plane $(a, b) \in \mathbb{R}^2$ the bifurcation curves where the roots of p, p' or p'' have multiple roots. We call *regions* the complement of these curves. The regions are open and connected sets in the parameter plane where the roots of p, p' and p'' are simple. In Sections 3, 4, 5 we study in turn the Newton's operator depending on the number of real roots of the polynomial p.

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2. Parameter Plane

As we pointed out, our main goal is to describe some primary (see below for a precise meaning of this) bifurcations occurring in the (real) parameter plane $(a, b) \in \mathbb{R}^2$ when considering the dynamical system given by the iterates of the Newton's method applied to the polynomial $p_{a,b}(x) = x^4 + a(x^3 + x) + bx^2 + 1$, denoted by $N_{a,b}$.

We firstly observe that we can restrict to (a, b)-parameters with $a \geq 0$ and that the Newton's method dynamical plane is symmetric under the transformation $z \mapsto \bar{z}$.

Lemma 1. *Let* $p_{a,b}(x) = x^4 + a(x^3 + x) + bx^2 + 1$ *and* $p_{-a,b}(x) = x^4 - a(x^3 + x) + bx^2 + 1$*, then* $N(p_{a,b})$ *is conjugate of* $N(p_{-a,b})$*. Moreover,* $N_{a,b}(\bar{z}) = \overline{N_{a,b}(z)}$*.*

Proof. We consider the map $\tau(x) = -x$, easy computations show that

$$N(p_{-a,b})(x) = (\tau^{-1} \circ N(p_{a,b}) \circ \tau)(x).$$

Using the expression of the Newton map (Eq. 2) and the fact that a and b are real numbers we have that $N_{a,b}(\overline{z}) = \overline{N_{a,b}(z)}$, so the dynamical plane is symmetric with respect to the real line. □

Primary bifurcations correspond to parameters $(a, b) \in \mathbb{R}^2$ for which the roots of p, p' or p'' collide. That is, the connected components of the complement of this set of parameters define regions in the parameter plane where the polynomials p, p', and p'' have simple roots. We notice, however, that in each of those regions the Newton's methods $N_{a,b}$ need not be, in general, dynamically equivalent. We will give precise results in this direction in next section; see, for instance, Proposition 1 or Proposition 2 and discussions therein.

We begin by studying the real and complex roots of $p(x)$. To obtain these roots we introduce a change of variable $y = x + \frac{1}{x}$. Then, the roots of $p(x)$ are given by

$$x = \frac{y_\pm \pm \sqrt{y_\pm^2 - 4}}{2}, \quad \text{where} \quad y_\pm = \frac{-a \pm \sqrt{a^2 - 4b + 8}}{2}.$$

From the study of these two discriminants we obtain the curves

$$\begin{aligned} L_1 &: = \{(a,b) \in \mathbb{R}^2 \mid a^2 - 4b + 8 = 0\}, \\ L_2 &: = \{(a,b) \in \mathbb{R}^2 \mid b = 2a - 2\}, \\ L_3 &: = \{(a,b) \in \mathbb{R}^2 \mid b = -2a - 2\}, \end{aligned}$$

separating the plane into different regions, depending on the number of real and complex roots contained in each of them.

There are one specific choice of the parameters, $(a,b) = (4,6)$, for which $p_{4,6}(x)$ has a unique root of multiplicity four. The bifurcations of the roots along the curves $L_1 \cup L_2 \cup L_3$ are of different nature. On the one hand, when the parameters $(a,b) \neq (4,6)$ are in L_1 the polynomial $p_{a,b}$ exhibits two roots of multiplicity two. These two double roots are complex for $0 \leq b < 4$ and are real for $b > 4$. On the other hand, when the parameters $(a,b) \neq (4,6)$ are in $L_2 \cup L_3$ the polynomial $p_{a,b}$ has three real roots, two simple and one double.

Secondly, we study the roots of $p'(x) = 4x^3 + 3ax^2 + 2bx + a$. The number of real roots of $p'(x)$ gives us the number of vertical asymptotes of the Newton's operator. Easily, on the curve

$$L_4 := \{(a,b) \in \mathbb{R}^2 \mid 27a^4 + 108a^2 - 108a^2 b - 9a^2 b^2 + 32b^3 = 0\}$$

$p'(x)$ has two real roots, one simple and one double. This curve delimits the regions where $p'(x)$ has one or three real roots.

Finally, the roots of $p''(x) = 12x^2 + 6ax + 2b$ are given by

$$c_1 = \frac{-3a - \sqrt{9a^2 - 24b}}{12}, \qquad c_2 = \frac{-3a + \sqrt{9a^2 - 24b}}{12}. \tag{3}$$

We observe that according to the used notation when c_1 and c_2 are real numbers we have that $c_1 < 0$ since $a > 0$, $c_1 \leq c_2$ and c_1 and c_2 collide on the curve

$$L_5 := \{(a,b) \in \mathbb{R}^2 \mid b = 3a^2/8\}.$$

From construction, the curves $L_1 \cup \cdots \cup L_5$ define the primary bifurcation parameters and the connected components of the complement in parameter plane

$$\{(a,b) \in \mathbb{R}^2 \,,\, a \geq 0\} \setminus \bigcup_{i=1}^{5} L_i,$$

which we denote by $\mathcal{R}$, are formed by parameter values where the polynomials p, p' and p'' have a constant number of simple roots. More precisely,

Remark 1. *Let $\mathcal{R}$ be a region. The number of real roots of p in $\mathcal{R}$ is 0, 2 or 4, the number of real roots of p' in $\mathcal{R}$ is 1 or 3, and the number of real roots of p'' in $\mathcal{R}$ is 0 or 2.*

Moreover, in $\mathcal{R}$ the roots of p, p' and p'' cannot collide themselves. Consequently, in $\mathcal{R}$, the roots of p cannot collide with roots of p', and the roots p'' cannot collide with roots of p'. However $\mathcal{R}$ can have parameters corresponding to collisions of the roots of p and p'' since those collisions do not imply the existence of a multiple root (see Lemma 2 below).

In the following table we summarize the number of real roots of p, p' and p'' in each region. To distinguish among regions we label them with a subscript.

Regions	*Roots of $p(x)$*	*Roots of $p'(x)$*	*Roots of $p''(x)$*
$\mathcal{R}_1$	2	1	2
$\mathcal{R}_2$	0	1	0
$\mathcal{R}_3$	0	1	2
$\mathcal{R}_4$	0	3	2
$\mathcal{R}_5$	4	3	2
$\mathcal{R}_6$	2	3	2
$\mathcal{R}_7$	2	1	0
$\mathcal{R}_8$	0	1	2
$\mathcal{R}_9$	0	3	2
$\mathcal{R}_{10}$	4	3	2
$\mathcal{R}_{11}$	2	3	2
$\mathcal{R}_{12}$	0	1	0

Table 1. Regions where the zeroes of p, p' and p'' are simple.

In Figure 2 we show the parameter plane (a,b), the bifurcation curves $L_1 \cup \cdots \cup L_5$ and the 12 regions.

Lemma 2. *The curves*

$$L_6 \;:\; = \{(a,b) \in \mathbb{R}^2 \mid 288 - 9a^4 + 3a\left(a^2 - 4b + 8\right)\sqrt{9a^2 - 24b} - 40b^2 + 24a^2(-3+2b) = 0\}$$
$$L_7 \;:\; = \{(a,b) \in \mathbb{R}^2 \mid 288 - 9a^4 - 3a\left(a^2 - 4b + 8\right)\sqrt{9a^2 - 24b} - 40b^2 + 24a^2(-3+2b) = 0\}$$

correspond to values of the parameters where the roots of p and p'' collide. Moreover, for $(a,b) \in L_6 \cup L_7$ the common root of p and p'' is an inflection point of the Newton operator.

Proof. The points w that are roots of p and p'' (but not roots of p') are the solutions of

$$p(w) = 0; \quad p'(w) \neq 0; \quad p''(w) = 0.$$

If the expression (3) of the critical points c_1 and c_2 is placed in the polynomial equation $p(x) = 0$ we obtain the expressions defining L_6 and L_7, respectively.

The second statement follows from direct computations, since $N'''(w) \neq 0$ and $N'(w) = N''(w) = 0$, as long as w satisfies $p(w) = p''(w) = 0$. So, w is an inflection point of the Newton operator. □

Next lemma is an auxiliary result that we will use in the study of the dynamics of the Newton's operator. However we state this result for a general iterated system.

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r) = r$.*

- *If $I = [r, x_0]$ and $r < f(x) < x$ for all $x \in (r, x_0]$, then $\lim_{n\to\infty} f^n(x) = r$ for all $x \in I$.*
- *If $I = [x_0, r]$ and $x < f(x) < r$ for all $x \in [x_0, r)$, then $\lim_{n\to\infty} f^n(x) = r$ for all $r \in I$.*

Proof. Let $x \in (r, x_0]$. We consider the iterates of x under f, i.e. the sequence of real numbers $f^n(x)$ for all $n \geq 0$. As, by hypothesis, $r < f(x) < x$ for all $x \in (r, x_0]$, we have that $r < f^{n+1}(x) = f(f^n(x)) < f^n(x)$ proving thus that the sequence $\{f^n(x)\}_{n\geq 0}$ is strictly decreasing. Since $r < f^n(x)$ for all $n \geq 0$ we conclude that the sequence of iterates converge to a limit point $\ell \geq r$. Using the continuity of f we have that $f(\ell) = \ell$ hence $\ell = r$.

The other item is proved by a similar reasoning. □

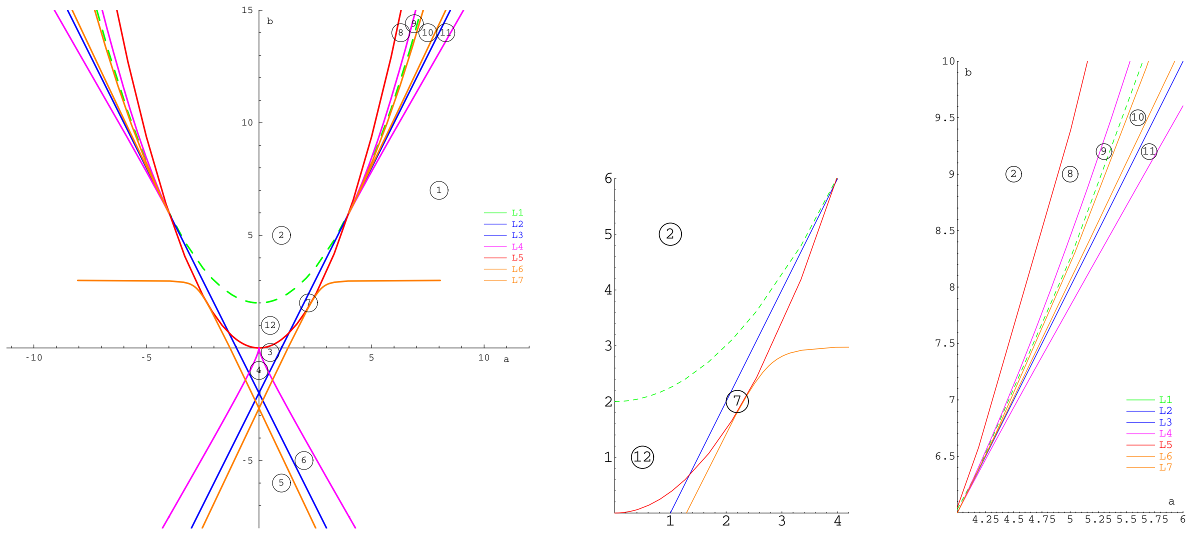


FIGURE 2. Regions in parameter plane bounded by the curves L_1, L_2, L_3, L_4 and L_5 and two zooms. The curves L_6 and L_7, corresponding to a collision of the roots of p and p'', are also drawn since they play an important role in the arguments.

3. THE POLYNOMIAL p HAS FOUR REAL ROOTS

In this section we assume that p has four real roots $r_1 < r_2 < r_3 < r_4$. Thus p' has exactly three real roots α_j, $j = 1, 2, 3$ satisfying $r_1 < \alpha_1 < r_2 < \alpha_2 < r_3 < \alpha_3 < r_4$, and p'' has exactly two real roots c_j, $j = 1, 2$ (that is, the two *free* critical points of $N_{a,b}$) satisfying $\alpha_1 < c_1 < \alpha_2 < c_2 < \alpha_3$; this case corresponds to regions $\mathcal{R}_5$ and $\mathcal{R}_{10}$. In Figure 3 we show the graph of $N_{a,b}$ for $(a, b) = (1, -8) \in \mathcal{R}_5$.

Proposition 1. *If the polynomial $p_{a,b}$ has four different real roots then, except for a measure zero set, all initial conditions converge to one of them, that is:*

$$\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4).$$

Proof. According to Table 1 the case when p has four real roots corresponds to regions $\mathcal{R}_5$ and $\mathcal{R}_{10}$. The difference between these two regions is the relative position of the roots respect to 0. Indeed, since $p(0) = 1$, in region $\mathcal{R}_5$ there are two negative and two positive roots while in the region $\mathcal{R}_{10}$ all the roots are negative.

In order to show that $\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4)$, it is enough to prove that the two free critical points c_1 and c_2 (see Eq. (3)) are *captured* by the roots of the polynomial $p_{a,b}$. Precisely, we should show that

$$\lim_{n\to\infty} N^n(c_2) = r_3 \text{ and } \lim_{n\to\infty} N^n(c_1) = r_2.$$

In fact we will show the first equality. The argument for the critical point c_1 is completely symmetric and details are left to the reader. The idea is to determine the behaviour of $N := N_{a,b}$ on the real line. The simple roots of p are fixed points of N. Moreover, N exhibits either a local maximum or minimum at simple, non common, zeros of p and p''. We also know that $y = \frac{3}{4}x - \frac{a}{16}$ is an oblique asymptote of N. With this in mind, we split the proof into two cases according to the relative position of the critical points and the roots of p, (notice that $c_2 \neq r_3$, otherwise we are done).

First we assume that $\alpha_2 < c_2 < r_3 < \alpha_3 < r_4$ and we claim that $\lim_{n\to\infty} N^n(c_2) = r_3$. To see the claim we first observe that N exhibits a local maximum at r_3 and a local minimum at c_2

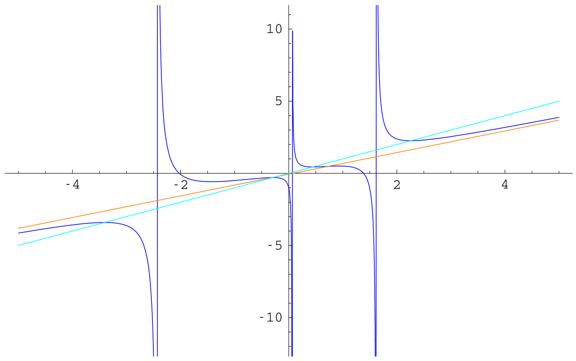


FIGURE 3. Graphics of Newton map $N_{-1,8}$ in region $\mathcal{R}_5$

and the mapping N is non decreasing in the interval $I = [c_2, r_3]$. Moreover, $x < N(x) < r_3$ for all $x \in [c_2, r_3)$. So Lemma 3 applies to conclude that $\lim_{n\to\infty} N^n(c_2) = r_3$.

Otherwise we assume that $\alpha_2 < r_3 < c_2 < \alpha_3 < r_4$ and we (also) claim that $\lim_{n\to\infty} N^n(c_2) = r_3$. It is easy to check that N exhibits a local maximum at c_2 and a local minimum at r_3. Moreover, $r_3 < N(x) < x$ for all $x \in (r_3, c_2]$. So, as above, Lemma 3 applies and $\lim_{n\to\infty} N^n(c_2) = r_3$.

□

4. THE POLYNOMIAL p HAS TWO REAL ROOTS

In this section we assume that p has two real roots $r_1 < r_2$. According to Table 1, this assumption corresponds to regions $\mathcal{R}_1, \mathcal{R}_6, \mathcal{R}_7$ and $\mathcal{R}_{11}$. In this case, depending on the region under consideration, the dynamics of $N_{a,b}$ can be from *simple* (that is, the *free* critical points are captured and the Fatou set coincide with the union of the attracting basins of the zeroes of $p_{a,b}$), see Proposition 2, to *rich* (that is either one or both of the *free* critical points are allowable to do their own dynamics and the Fatou set is not reduced to points whose orbits converge to one of the roots of $p_{a,b}$), see Proposition 3. We exemplify the later case by means of some numerical experiments.

Proposition 2. *Let $(a, b) \in \mathcal{R}_7$. Then, except for a measure zero set, all initial conditions converge to one of the roots of p, that is:*

$$\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4).$$

Proof. In this case p has two real roots $r_1 < r_2$ and two complex (conjugated) roots r_3 and $r_4 = \overline{r_3}$, p' one real root α and two complex (conjugated) roots α_2 and $\alpha_3 = \overline{\alpha_2}$, and p'' has two complex (conjugated) roots c_1 and $c_2 = \overline{c_1}$. We assume that $\mathrm{Im}(r_3) > 0$, $\mathrm{Im}(\alpha_2) > 0$ and $\mathrm{Im}(c_1) > 0$. Moreover the Julia and Fatou sets of $N_{a,b}$ are symmetric with respect to the real line, since p has real coefficients (see Lemma 1).

The idea of the proof is the following. Since the Julia set of $N_{a,b}$ is a connected set of the Riemann sphere $\hat{\mathbb{C}}$ we know that the four immediate basins of attraction of r_i for $i = 1, 2, 3, 4$ are simply connected and we will study the degree of $N_{a,b}$ in every immediate basin of attraction, proving that in fact $N_{a,b} : A^*(r_2) \to A^*(r_2)$ has degree four. This situation forces the two critical points c_1 and c_2 to belong to it. So, $\lim_{n\to\infty} N^n_{a,b}(c_i) = r_2$ for $i = 1, 2$.

There is a unique configuration for the real roots of p and p' given by $r_1 < \alpha < r_2$, so N has a local maximum at $x = r_1$ and a local minimum at $x = r_2$. From Lemma 3, we have that

$\forall\, x_0 \in (-\infty, r_1)$ then $\lim_{n\to\infty} N^n(x_0) = r_1$. On the other hand, $\forall\, x_0 \in (r_1, \alpha)$, as r_1 is a local maximum, $N(x_0) < r_1$, that is, $N(x_0) \in (-\infty, r_1)$ and therefore Lemma 3 is also applied. So,

$$\lim_{n\to\infty} N^n(x_0) = r_1, \quad \forall\, x_0 \in (-\infty, \alpha).$$

Similarly, we have that

$$\lim_{n\to\infty} N^n(x_0) = r_2, \quad \forall\, x_0 \in (\alpha, \infty).$$

In particular, $A^*(r_1) \cap \mathbb{R} = (-\infty, \alpha)$, $A^*(r_2) \cap \mathbb{R} = (\alpha, +\infty)$ and $N_{a,b}(\alpha) = \infty$. Thus, $A^*(r_3)$ is contained in the upper half plane and the $A^*(r_4)$ is contained in the lower half plane.

We recall that $N_{a,b}$ is a rational map defined on the Riemann sphere $\hat{\mathbb{C}}$ with degree 4. Let $\Gamma = (-\infty, r_1] \cup (r_2, \infty) \cup \{\infty\}$ and consider $N_{a,b}^{-1}(\Gamma)$. This set consists into four pieces: the first one is Γ itself, the second one is $[r_1, r_2]$ and the two other preimages are two curves γ_1 and γ_2 containing α_2 and α_3, respectively. We notice that these two curves are symmetric respect to the real line. Thus, they divide the complex plane into two doubly connected components

$$A_+ = \{z \in \mathbb{C};\ Im(z) > 0\} \setminus \gamma_1 \text{ and } A_- = \{z \in \mathbb{C};\ Im(z) < 0\} \setminus \gamma_2.$$

By construction $N_{a,b}$ maps $A_\pm$ onto $\hat{\mathbb{C}} \setminus \Gamma$. Using the Riemann-Hurwitz formula (see [Mil06]) we have that every point in $\hat{\mathbb{C}} \setminus \Gamma$ has exactly two preimages in A_+ and two preimages in A_-. We consider now the immediate basin of attraction of r_3 denoted by $A^*(r_3)$, by construction we have that $A^*(r_3) \subset A_+$, since $A^*(r_3)$ cannot cross the real line and $r_3 \in A_+$. Thus, the other two preimages of $A^*(r_3)$ belong to A_-. More precisely one preimage attached to α and the other attached to α_3. By symmetry, $A^*(r_4) \subset A_-$ and the other two preimages belong to A_+, one attached to α and the other attached to α_2. Concluding thus that $N_{a,b} : A^*(r_i) \to A^*(r_i)$ has degree 2 for $i = 3, 4$.

We claim that Newton's map has degree four in $A^*(r_2)$. To see the claim we assume that $N_{a,b} : A^*(r_2) \to A^*(r_2)$ has degree two. In this case the dynamics of $N_{a,b}$ is conformally conjugate to $z \to z^2$ defined in the unit disc and we can define internal rays of angle θ in the immediate basin. These rays are defined as the image, under the conformal conjugacy, of the segment $re^{i\theta}$ for $0 \le r < 1$. The conjugacy $z \to z^2$ induces a conjugacy between the internal rays $\theta \to 2\theta$. Since α is a preimage of ∞ by construction at least three rays land at α. One of these three rays is $[\alpha, r_2]$ and the other two rays are symmetric respect to the real line separating $A^*(r_2)$ from the preimages of $A^*(r_3)$ and $A^*(r_4)$ attached to it. These three rays are mapped under the doubling map to the fixed ray $[r_2, \infty)$. This is a contradiction with the fact that the map has degree two, since the only preimage of the angle 0 under the doubling map $\theta \to 2\theta$ is the unique angle $\theta = 1/2$.

For parameter values (a, b) with $a < 0$ we obtain the symmetric situation, i.e. the degree of Newton's map in $A^*(r_1)$ is four. □

Proposition 3. *Let assume $(a, b) \in \mathcal{R}_1 \cup \mathcal{R}_6$. Then $c_1 \in A(r_2)$.*

Proof. Let assume $(a, b) \in \mathcal{R}_1$. In this region the polynomial $p := p_{a,b}$ has two real roots $r_1 < r_2$, the polynomial p' has only one real root α, and the polynomial p'' has two real roots $c_1 < c_2$. We obviously have that $r_1 < \alpha < r_2$. If $c_1 = r_2$ the statement is trivial. So, in what follow we assume $c_1 \neq r_2$.

Taking a concrete value of the parameter in $\mathcal{R}_1$ it is easy to check that $r_1 < \alpha < r_2 < c_1 < c_2$. Consequently, for all parameters in $\mathcal{R}_1$ we have $\alpha < c_1$, since otherwise we should be crossing a bifurcation curve. Hence there are only three possible configurations depending on the relative positions of the points c_1, c_2 and r_2 which indeed correspond to the three connected components of $\mathcal{R}_1 \setminus L_6$.

Case (a). $r_1 < \alpha < r_2 < c_1 < c_2$. The Newton's map N has a local minimum at r_2 and a local maximum at c_1. Defining $I = [r_2, c_1]$ we have that $r_2 < N(x) < x$ for all $x \in (r_2, c_1]$ and Lemma 3 concludes that $\lim_{n\to\infty} N^n(c_1) = r_2$.

Case (b). $r_1 < \alpha < c_1 < r_2 < c_2$. The Newton's map exhibits a local minimum at c_1 and a local maximum at r_2. Taking $I = [c_1, r_2]$ we have that $x < N(x) < r_2$ for all $x \in [c_1, r_2)$ and applying Lemma 3 we conclude that $\lim_{n\to\infty} N^n(c_1) = r_2$.

Case (c). $r_1 < \alpha < c_1 < c_2 < r_2$. In fact under this configuration we will show that $c_i \in A(r_2), i = 1, 2$.

The Newton's map has local minima at r_2 and c_1, and local maxima at c_2 and r_1. Since $r_2 < N(x) < x$ for all $x \in (r_2, +\infty)$, Lemma 3 implies that $N^n(x) \to r_2$ for all $x > r_2$. The first iterate $N(c_2) \in (r_2, +\infty)$, and then $\lim_{n\to\infty} N^n(c_2) = r_2$. This proves $c_2 \in A(r_2)$.

On the other hand, N is increasing in (c_1, c_2). If $N(c_1) \geq c_2$ then $\lim_{n\to\infty} N^n(c_1) = r_2$ and we are done. So, we assume that $N(c_1) < c_2$. We claim that there exists a minimal index $k > 0$ such that $N^k(c_1) > c_2$. To see the claim, we assume that $N^k(c_1) \leq c_2$ for all $k \geq 0$, so this increasing sequence is bounded and has a limit ℓ which is a fixed point of N in the interval $(c_1, c_2]$ which is a contradiction.

To finish the proof we deal with $(a, b) \in \mathcal{R}_6$. In this region the polynomial $p := p_{a,b}$ has two real roots $r_1 < r_2$, the polynomial p' has three roots $\alpha_1 < \alpha_2 < \alpha_3$, and the polynomial p'' has two real roots $c_1 < c_2$. Arguing as before, since there are no multiple zeroes of the polynomials p, p' and p'' in this region, we take one concrete parameter in $\mathcal{R}_6$ to show that for all parameters in $\mathcal{R}_6$ we have

$$r_1 < \alpha_1 < r_2 < \alpha_2 < c_2 < \alpha_3.$$

Considering the value of c_1, there are only two possible configurations: $c_1 < r_2$ or $r_2 < c_1$, which correspond to the two connected components of $\mathcal{R}_6 \setminus L_6$.

Case (a). $r_1 < \alpha_1 < c_1 < r_2 < \alpha_2 < c_2 < \alpha_3$. The Newton's map exhibits a local minimum at c_1 and a local maximum at r_2. Taking $I = [c_1, r_2]$ we have that $x < N(x) < r_2$ for all $x \in [c_1, r_2)$ and applying Lemma 3 we conclude that $\lim_{n\to\infty} N^n(c_1) = r_2$.

Case (b). $r_1 < \alpha_1 < r_2 < c_1 < \alpha_2 < c_2 < \alpha_3$. Now, the Newton's map exhibits a local minimum at r_2 and a local maximum at c_1. Defining $I = [r_2, c_1]$ we have that $r_2 < N(x) < x$ for all $x \in (r_2, c_1]$ and by Lemma 3 we conclude that $\lim_{n\to\infty} N^n(c_1) = r_2$.

So the Proposition follows. $\square$

4.1. **Further numerical experiments.** From the above results we know that for most of initial conditions $N_{a,b}$, $(a, b) \in \mathcal{R}_7$ orbits converge to one of the roots of p. In other words, for every parameter in $\mathcal{R}_7$, except for a measure zero set of initial conditions, the Fatou set is the union of the basins of the roots of p.

In $\mathcal{R}_1$ and $\mathcal{R}_6$ the situation is quite different. Although the smallest free critical point, c_1, is captured by r_2 we still do not know in general the dynamics of the critical point c_2. In fact, the following numerical examples illustrate that, for some parameters, there are open sets of initial conditions (in the dynamical plane) where the orbit do not converge to any of the roots. Moreover, by means of the Implicit Function Theorem, those *bad* parameter values form an open set in parameter plane.

Example 1. *Let* $(a, b) = (1.45577, -2.10306) \in \mathcal{R}_1$. *The (biggest, real) critical point of* $N_{a,b}$ *is* $c_2 \approx 0.3310137$. *Doing some numerics it is possible to show that the sequence* $N^n_{a,b}(c_2)$ *is* *attracted by the attracting 2-cycle (see Figure 4(a))*

$$\{\eta_1 \approx -1.5846742, \eta_2 \approx 0.33143177\}.$$

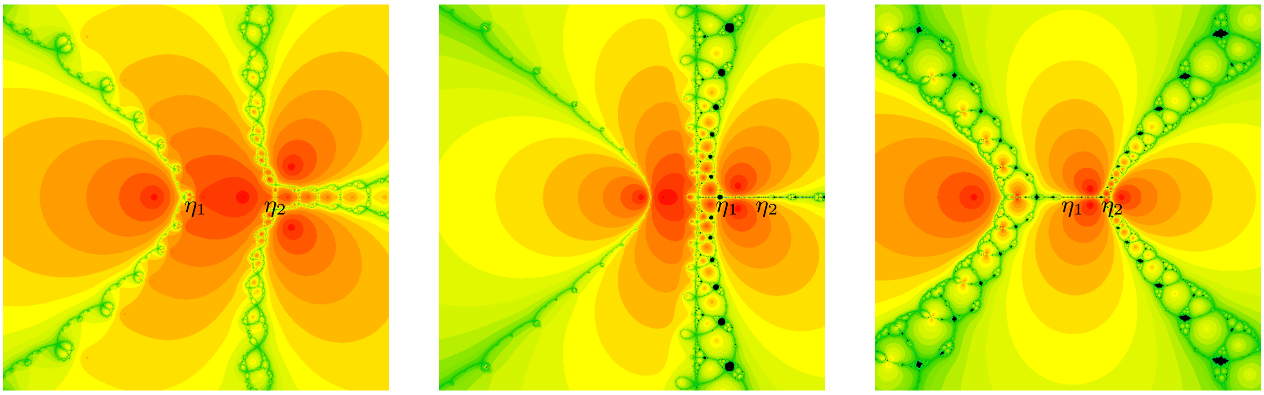
Example 2. *Let* $(a, b) = (0.129845, -1.99029) \in \mathcal{R}_6$*. The (biggest, real) critical point of* $N_{a,b}$ *is* $c_2 \approx 0.5444$*. Doing some numerics it is possible to show that the sequence* $N^n_{a,b}(c_2)$ *is attracted by the attracting 2-cycle (see Figure 4(b))*

$$\{\eta_1 \approx 0.557, \eta_2 \approx 1.008\}.$$

Example 3. *In case of region* $\mathcal{R}_{11}$ *the polynomials* p *and* p'' *has two real roots and* p' *has three real roots. The relative configuration of those roots is unique:* $r_1 < \alpha_1 < c_1 < \alpha_2 < c_2 < \alpha_3 < r_2$*. The following numerical example shows that there are parameter values (in fact, as before, open sets of them in parameter plane) for which none of the free critical points is captured by any of the attracting basins of the roots of* p*.*

Let $(a, b) = (5.93336, 9.80305) \in \mathcal{R}_{11}$*. The real critical points of* $N_{a,b}$ *are* $c_1 \approx -2.236$ *and* $c_2 = -0.7307$*. The sequence* $N^n_{a,b}(c_i)$*,* $i = 1, 2$ *is attracted by the attracting 2-cycle (see Figure 4(c))*

$$\{\eta_1 \approx -1.0233, \eta_2 \approx -0.7199\}.$$



(a) Parameters in $\mathcal{R}_1$ (b) Parameters in $\mathcal{R}_6$ (c) Parameters in $\mathcal{R}_{11}$

FIGURE 4. Dynamical planes for parameters in $\mathcal{R}_1$, $\mathcal{R}_6$ and $\mathcal{R}_{11}$. In all three cases the Fatou set is different from the union of all the basin of attraction of the roots of p.

5. p HAS NO REAL ROOTS

According to Table 1, we consider parameter values in $\mathcal{R}_i$ with $i \in \{2, 3, 4, 8, 9, 12\}$. In all these regions we will show that the orbit of the two *free* critical points, c_1 and c_2 need not be captured by the roots of the polynomial p. So, in some cases, the Fatou set will be larger than the union of the attracting basins of the roots of p. Our main goal is to give concrete possible scenarios and configurations in the dynamical plane.

The main argument is to study in detail the bifurcation curve L_5. We recall that the bifurcation curve $L_5 = \{(a, b) \in \mathbb{R}^2 \,|, b = 3a^2/8\}$ consists of the set of parameter values for which the two simple critical points c_1 and c_2 collide in a double critical point located at $c_1 = c_2 = -a/4$. Under this assumption we can treat the Newton's map

$$\mathcal{N}_a := N_{a,b} \qquad \text{where} \qquad b = \frac{3a^2}{8}$$

as a one *complex* parameter family of rational maps in the Riemann sphere. By construction $\mathcal{N}_a$ has a unique *free* critical point of multiplicity two located at $-a/4$. Moreover,

$$\mathcal{N}_a(z) = z - \frac{p_{a,3a^2/8}(z)}{p'_{a,3a^2/8}(z)} = z - \frac{z^4 + a(z^3+z) + \frac{3}{8}a^2z^2 + 1}{4z^3 + a(3z^2+1) + \frac{3}{4}a^2 z}, \quad a \in \mathbb{C}$$

depends holomorphically on a.

Remark 2. *We notice that studying $\mathcal{N}_a$ on the real line we are studying $N_{a,b}$ on the bifurcation parabola $b = 3a^2/8$ in the $(a,b)-$parameter plane. In Figure 5 (a) and (b) we plot the parameter plane of $\mathcal{N}_a$.*

We first state that the $a-$plane has two symmetries. The proof is direct.

Lemma 4. *Let $a \in \mathbb{C}$. Then $\mathcal{N}_a(z)$ is conjugate to $\mathcal{N}_{\overline{a}}(z)$ and $\mathcal{N}_{-\overline{a}}(z)$. Hence it is enough to study the closed first quadrant.*

Following [McM00], it is clear that our family is non-trivial, that is the *bifurcation locus*, given by the parameter values for which the number of attracting cycles is not locally constant, is non empty. Hence, it follows from Theorem 1.1 and Corollary 1.2 in [McM00] and taking into account that the critical point $z = -a/4$ has multiplicity two, that our bifurcation locus contains, densely, copies of small copies of the generalized Mandelbrot set $\mathcal{M}_3$, defined as

$$\mathcal{M}_3 = \{c \in \mathbb{C}\,,\, \mathcal{J}(z^3+c) \text{ is connected }\}.$$

To illustrate this claim we plot in Figure 5(c) the parameter plane of $z^3 + c$ and in Figure 5 (b) a small copy of $\mathcal{M}_3$ in the parameter plane of $\mathcal{N}_a$. For the hyperbolic a-parameter values belonging to the interior of these small copies of $\mathcal{M}_3$ the critical point is attracted by an attracting k-cycle, $k \geq 2$, and hence its dynamics is not captured by the attracting basins of the roots of p. In other words, for all these *bad* parameters the Fatou set is larger than the union of the attracting basins of the roots of p.

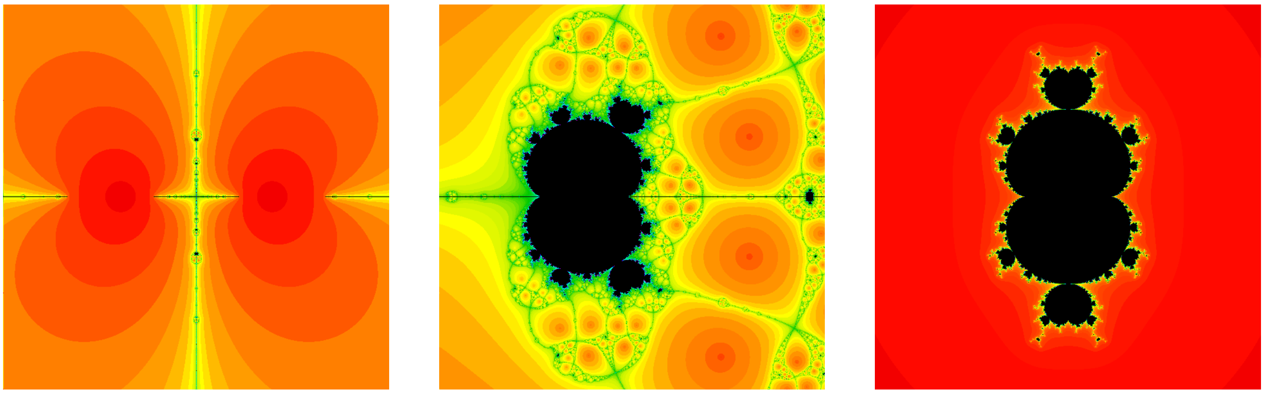
Remark 3. *It follows from the symmetries described in Lemma 1 that those small copies of $\mathcal{M}_3$ in the parameter plane of $\mathcal{N}_a$ are symmetrically located with respect the real (and complex) line. Moreover, it is possible to find numerically some real parameters $a \in \mathbb{R}$ such that the critical point $-a/4$ belongs a superattracting cycle of period $k \geq 2$ (at least for some values of k) since such parameters are real solutions of the equation $G_k(a) = 0$, where*

$$G_k(a) = \mathcal{N}_a^k(-\frac{a}{4}) + \frac{a}{4} = 0.$$

Consequently, infinitely many of them, of arbitrarily high period, cross the real line, and are symmetric with respect to it. See again Figure 5(a-b). Numerical computations show that for $a_1 = 1.1047321$ the map $\mathcal{N}_{a_1}$ exhibits a superattracting cycle at $\{-0.276183, -0.9742426\}$ and for $a_2 = 4.24023195$ and $\mathcal{N}_{a_2}$ a superattracting cycle at $\{-0.5008507, -1.060058\}$. We notice that a_1 and a_2 correspond to real solutions of $G_2(a) = 0$.

Remark 2 implies that $(a,b) = (a_1, \frac{3}{8}a_1^2)$ and $(a,b) = (a_2, \frac{3}{8}a_2^2)$ correspond to parameter values in L_5 having an attracting cycle of period two (and so the *free* critical point belongs to its immediate basin). These are only two points in L_5 but we know from [McM00] that those hyperbolic parameters are dense in the bifurcation locus intersection with $\mathbb{R}$, and so in the corresponding pieces of L_5.

Set $(\hat{a}, \frac{3}{8}\hat{a}^2) \in L_5$ a parameter in the (a,b)-plane such that $\hat{a} \in \mathbb{R}$ is a (real) parameter belonging to a small copy of $\mathcal{M}_3$. Consequently, for this parameter the dynamical plane has an attracting k-cycle ($k \geq 2$) and the free critical point belongs to its immediate basin. Applying the Implicit Function Theorem we deduce that for (a,b)-parameters in a sufficiently small neighbourhood of $(\hat{a}, \frac{3}{8}\hat{a}^2)$ the same dynamical behaviour occurs and the continuation of the attracting k-cycle should remain. As a corollary, we know that for all the regions $\mathcal{R}$ bounded



(a) Parameter plane of $\mathcal{N}_a$. (b) Zoom of (a). (c) Parameter plane of $z^3 + c$.

FIGURE 5. The parameter plane of $\mathcal{N}_a$ for complex values of a (figures (a) and (b)) and the parameter plane of $z^3 + c$. (figure (c)).

by L_5 (see Figure 2) we will find parameters for which the Fatou set contains open sets of initial conditions which does not converge to any of the roots of p.

However, we have proven in Proposition 2 that in $\mathcal{R}_7$ all parameters satisfies that

$$\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4).$$

Hence, the piece of L_5 in the boundary of $\mathcal{R}_7$ should correspond precisely to the piece of $\mathbb{R}$ in the a-plane which does not intersect the bifurcation locus. We have proven that

Proposition 4. *There are parameters values $(a, b) \in \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_8, \mathcal{R}_{12}$ such that*

$$A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4) \subsetneq \mathcal{F}(N_{a,b}).$$

5.1. **Further numerical experiments.**

We finish the study of the (a, b)-parameter plane with the the remainder regions: $\mathcal{R}_4$ and $\mathcal{R}_9$. In both cases the polynomial p has no real roots while p'' has two real roots. Since the polynomial p has real coefficients, the real line is forward invariant. Thus, on the one hand neither c_1 nor c_2 could tend under iteration to any root of p, and on the other hand none of the basins of attraction of the roots of p intersect the real line. Moreover, Lemma 1 concludes that the Fatou and Julia sets of $N_{a,b}$ need to be symmetric with respect to the real and imaginary line. Above arguments imply that the *free* (real) critical points, $c_1 < c_2$, cannot be captured by the basin of attractions of the root of p. Next numerical examples show that in fact each of the *free* critical points may have it own dynamics.

Example 4. *Let $(a, b) = (0.0013, -1.7373) \in \mathcal{R}_4$. The critical points of $N_{a,b}$ are $c_1 \approx -0.5384$ and $c_2 \approx 0.5378$. The sequence $N^n_{a,b}(c_1)$ is attracted by the attracting 2-cycle*

$$\{\eta_1 \approx -0.52347, \eta_2 \approx -1.003\},$$

while $N^n_{a,b}(c_2)$ is attracted by the attracting 2-cycle

$$\{\nu_1 \approx 0.54304, \nu_2 \approx 1.0058\}.$$

In Figure 1 we illustrate this dynamical plane.

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