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Integrability and Algebraic Solutions for Planar Polynomial Differential Systems with Emphasis on the Quadratic Systems

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Abstract: The paper is divided into two parts. In the first one we present a survey about the theory of Darboux for the integrability of polynomial differential equations. In the second part we apply all mentioned results on Darboux theory to study the integrability of real quadratic systems having an invariant conic. The fact that two intersecting straight lines or two parallel straight lines are particular cases of conics allows us to study simultaneously the integrability of quadratic systems having at least two invariant straight lines.

Key words: Integrability, quadratic differential systems, algebraic solutions.

1. Introduction

By definition a *polynomial system* is a differential system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), \tag{1}$$

where P and Q are polynomials with coefficients in $\mathbf{F}$, where $\mathbf{F}$ will denote either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. The polynomial systems of degree 2 will be called *quadratic systems*. In this paper we only consider polynomial systems (1) such that P and Q are relatively prime. In other words, we only consider polynomial systems (1) having finitely many singular points. This work contributes to show the link between the theories of polynomial systems and algebraic curves. Indeed, already in 1878, Darboux [7] showed how the first integrals of polynomial systems possessing sufficient algebraic solutions are constructed (see Darboux Theorem in Section 2). In particular, he proved that if a polynomial system of degree m has at least $m(m+1)/2$ algebraic solutions, then it has a first integral. On the other hand such links were also suggested in 1900 by the way that Hilbert [12] stated his 16th problem, in two parts: the first one about the topology of real algebraic curves and the second one about the maximum number of limit cycles of polynomial systems having a given degree. Recently, this link appeared in the theory of the center for quadratic systems. See the work of Schlomiuk [27, 28, 29], and in particular the Theorem of Schlomiuk–Guckenheimer–Rand, in Section 3, and our Theorem 4. But this link is not restricted to quadratic systems and there is a wide literature on this question in recent years; a general reference is the paper of Pearson, Lloyd and Christopher [22].

In the first part of this paper (Sections 2 and 3) we present a short survey about the Darboux theory of integrability; in the second part (Sections 4 and 5)