Integrability and Alegbraic Solutions for the 2-D Lotka–Volterra System

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Abstract. We apply the method of Darboux to study the integrability of the 2d Lotka-Volterra system. The method of Darboux, which consists of finding first integrals by using algebraic solutions, is reduced in the case of the Lotka-Volterra system to obtaining only one algebraic solution different from the two axes, which can be a polynomial of arbitrary degree. Here we are concerned with polynomials of degree less than five. We find that all the essential parameters of the Lotka-Volterra system must be fixed for the existence of a polynomial solution of degree greater than or equal to three whereas, for degrees one and two respectively, only one and two conditions on the parameters are required.

1 Introduction

The Lotka-Volterra system in two dimensions (2-d) is a particular quadratic system of the form

$$\dot{x} = x(ax + by + c), \qquad \dot{y} = y(Ax + By + C). \tag{1}$$

This system and its generalisation to n-d are called *Lotka-Volterra systems* (LVS) because they were initially studied by Lotka (1920) and Volterra (1931). Later Kolmogorov (1936) also studied these systems and for this reason some authors denote them by *Kolmogorov systems*. There are many natural phenomena that can be modeled by the LVS such as the time evolution of conflicting species in biology (May 1974), chemical reactions, plasma physics (Laval and Pellat 1975), hydrodynamics (Busse 1978), economics etc. Starting with Volterra (1931) mathematicians have been concerned particularly with the problem of integrability, which is only possible for a very restricted set of parameters. Cairó and Feix (1992) used a generalised Carleman method (Carleman 1932) to obtain some integrable cases, generalising the result to the n-d version of (1). This was facilitated by the fact that here $x = 0$ and $y = 0$ are invariant, thereby linearising the problem. However, for a general quadratic system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{2}$$