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Darboux integrability for 3D Lotka–Volterra systems

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Abstract. We describe the improved Darboux theory of integrability for polynomial ordinary differential equations in three dimensions. Using this theory and computer algebra, we study the existence of first integrals for the three-dimensional Lotka–Volterra systems. Only working up to degree two with the invariant algebraic surfaces and the exponential factors, we find the major part of the known first integrals for such systems, and in addition we find three new classes of integrability. The method used is of general interest and can be applied to any polynomial ordinary differential equations in arbitrary dimension.

1. Introduction

Nonlinear ordinary differential equations appear in many branches of applied mathematics and physics. In dimensions greater than two these systems usually present chaotic motion in the sense that they depend sensitively on the choice of initial conditions, and more specifically the difference between the initial conditions grows exponentially with time. It is important to find conditions for the absence of this chaotic motion by looking for parameter values for which the systems can be partially or completely integrable. For three-dimensional systems the existence of one first integral means that the system is partially integrable, and the existence of two independent first integrals means that the system is completely integrable (because the phase portrait is then completely characterized). If a three-dimensional system is integrable its solutions have good behaviour and it is possible to obtain global information on its long-term evolution. Since the notion of integrability is based on the existence of first integrals the following natural question arises. Given a system of ordinary differential equations depending on parameters, how does one recognize the values of the parameters for which the system has first integrals? Many different methods have been used to study the existence of first integrals. Some of them have been developed for Hamiltonian systems, such as the Ziglin [1, 2] analysis, or the method based on the Noether symmetries [3]. Other methods can be applied to non-Hamiltonian systems: the method of Darboux [4], the method of Lie symmetries [5], the Painlevé analysis [6], the use of Lax pairs [7], the direct method [8], the linear compatibility analysis method [9], the Carlemann embedding procedure [10, 11], the quasimonomial formalism [12], etc.

In 1878 Darboux [4] showed how one could construct the first integrals of planar polynomial ordinary differential equations possessing sufficient invariant algebraic curves. In particular, he proved that if a planar polynomial ordinary differential system of degree m

