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Integrability of the 2D Lotka–Volterra system via polynomial first integrals and polynomial inverse integrating factors

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Abstract. We present new first integrals of the two-dimensional Lotka–Volterra systems which have a polynomial inverse integrating factor. Moreover, we characterize all the polynomial first integrals of the two-dimensional Lotka–Volterra systems.

1. Introduction and statement of the results

The two-dimensional Lotka–Volterra dynamical system is defined by

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y) \qquad \dot{y} = y(a_2 + b_{21}x + b_{22}y) \tag{1}$$

where $(a_1, b_{11}, b_{12}, a_2, b_{21}, b_{22})$ are six real (or complex) parameters. This system introduced by Lotka [1] and Volterra [2] appears in chemistry and ecology where it models two species in competition, and it has been widely used in applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, convective instabilities, neural networks, etc. Many authors have examined the integrability of the two-dimensional Lotka–Volterra systems, see for instance Cairó *et al* [3] (who used the Carleman method), Hua *et al* [4] (who used the Hamiltonian method), Cairó and Llibre [5] and Cairó *et al* [6] (who used the Darboux theory of integrability) or the integrability of the three-dimensional Lotka–Volterra systems (see Grammaticos *et al* [7], Almeida *et al* [8] and Cairó and Llibre [9]). These systems have been studied in arbitrary dimension by Cairó *et al* [3] and Cairó and Feix [10].

Recently, Moulin-Ollagnier [11] and Labrunie [12] have characterized the polynomial first integrals of a special three-dimensional Lotka–Volterra system, the so-called ABC system, i.e.

$$\dot{x} = x(Cy + z) \qquad \dot{y} = y(x + Az) \qquad \dot{z} = z(Bx + y).$$

The fact that the vector fields associated with the ABC systems are homogeneous helps in the study of their polynomial first integrals. In general, this is not the case for system (1), but of course this system is simpler than the ABC system as it is two dimensional.

The problem of the integrability of ordinary differential equations is closely related to the problem of finding first integrals. The difficulty of the task was already noted by Poincaré [13] in his discussion of a method to obtain polynomial or rational first integrals. The search

