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SINGULAR PERTURBATIONS OF BLASCHKE PRODUCTS AND CONNECTIVITY OF FATOU COMPONENTS

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Abstract. The goal of this paper is to study the family of singular perturbations of Blaschke products given by $B_{a,\lambda}(z) = z^3 \frac{z-a}{1-\overline{a}z} + \frac{\lambda}{z^2}$. We focus on the study of these rational maps for parameters a in the punctured disk $\mathbb{D}^*$ and $|\lambda|$ small. We prove that, under certain conditions, all Fatou components of a singularly perturbed Blaschke product $B_{a,\lambda}$ have finite connectivity but there are components of arbitrarily large connectivity within its dynamical plane. Under the same conditions we prove that the Julia set is the union of countably many Cantor sets of quasicircles and uncountably many point components.

1. **Introduction.** Given a Rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere, the Fatou set $\mathcal{F}(f)$ is defined as the set of points $z \in \widehat{\mathbb{C}}$ such that the family of iterates $\{f(z), f^2(z) = f(f(z)), ...\}$ is normal in some open neighbourhood of z. Its complement, the Julia set $\mathcal{J}(f)$, corresponds to the set of points where the dynamics is chaotic. The Fatou and the Julia sets are totally invariant under $f(z)$. The Fatou set is open and its connected components, known as Fatou components, map among themselves. The celebrated result of Sullivan [18] states that all Fatou components of rational maps are either periodic or preperiodic. Moreover, any cycle of periodic Fatou components has at least a critical point, i.e. a zero of $f'(z)$, somehow related to it (see [11]).

The aim of this paper is to study singular perturbations of a family of Blaschke products and analyse the structure of their dynamical plane. We focus on the special case for which Fatou components of arbitrarily large connectivity appear. The study of singular perturbations of rational maps is a very active research field in holomorphic dynamics. They were used by McMullen to show the existence of buried Julia components for rational maps, i.e. connected components of the Julia set which do not intersect the boundary of any Fatou component. He described

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