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New Results in Averaging Theory and its Applications

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Chapter 1

Introduction and statements of the results

The averaging theory basically consists in replacing a vector field

$$x' = F(t, x, \varepsilon), \text{ with } (t, x, \varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0),$$

by its average over the time or over an angular variable with the goal to obtain asymptotic approximations to the solutions of the original system and to obtain periodic solutions. Although this theory was originated in the $18th$ century, until 1928 it was not proved rigorously by Fatou(see [30]).

The averaging theory for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions in a vector field by studying the equilibrium points of its associated averaged system.

This theory becomes a classical tool for studying periodic solutions of nonlinear differential systems, see for instance [28, 56, 64, 67, 86]. Moreover, remarkable contributions to it were made by Krylov and Bogoliubov [45] in the 1930s and Bogoliubov [5] in 1945. For a brief historical review, the interested reader is referred to [68, Appendix A].

In this work we will improve the averaging theory for finding periodic solutions. Then we will propose a method for studying the stability of periodic solutions that are non linearly hyperbolic. Finally, using these new results we present several applications of the theory. In particular we shall apply the new theoretical result here presented to differential systems that could not be studied with the classical results.

The system $x' = F(t, x, 0)$ is called the *unperturbed system*. Concerning the averaging theory for finding limit cycles, two main hypotheses are usually assumed: (i) F is T-periodic in the first variable; and (ii) there exists a sub-manifold $\mathcal{W} \subset \mathbb{R}^n$ such that each solution of the unperturbed system with initial condition in $\mathcal{W}$ is T-periodic. Under these hypotheses the averaging theory provides sufficient conditions for the existence of limit cycles of $x' = F(t, x, \varepsilon)$.

The classical averaging theorem for the existence of limit cycles can be stated as follows. Consider the initial value problem

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 \widetilde{F}(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

and

$$\dot{\mathbf{y}} = \varepsilon g_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{1.2}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\boldsymbol{x_0}$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume that F_1, $\widetilde{F}$ are T–periodic in the variable t, and we set

$$g_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})dt. \tag{1.3}$$

Theorem 1. *Assume that F_1, $\widetilde{F}$, $D_x F_1$, $D_{xx} F_1$ and $D_x \widetilde{F}$ are continuous and bounded by a constant M independent of ε in $[0, \infty) \times \Omega \times [-\varepsilon_0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/|\varepsilon|]$. Then the following statements hold:*
(a) For $t \in [0, 1/|\varepsilon|]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.
(b) If $\boldsymbol{s}$ is a singular point of system (1.2) and $\det D_{\mathbf{y}} g_1(\boldsymbol{s}) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ for system (1.1) which is close to $\boldsymbol{s}$ and such that $\varphi(0, \varepsilon) - \boldsymbol{s} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.
(c) The stability of the periodic solution $\varphi(t, \varepsilon)$ is given by the stability of the singular point.

For a proof of Theorem 1 see [81, Theorem 11.5], where it is stated on the $\varepsilon \in [0, \varepsilon_0)$ but in fact following the proof the same result works for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ as it is stated here.

In the last decade this theory has increased immensely. Several works have been dedicated to extend the averaging theory to a wider class of differential systems. For instance, in [11], taking advantage of the Browder degree theory, it was developed a topological version of the first-order averaging method to study the existence of limit cycles in continuous vector fields. Their stability properties were investigated in [7], and in [54] topological version of the averaging method was extended at any order. The averaging theory has also been considered in a discontinuous context. For instance, in [54, 50], the averaging method was developed up to order 2 for discontinuous differential system, and in [40, 52] the averaging method was extend at any order for a class of discontinuous differential system.

The first result here presented (see Theorem 2) provides sufficient conditions to assure the persistence of some zeros of smooth functions $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ having the form

$$g(z, \varepsilon) = g_0(z) + \sum_{i=1}^{k} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}). \tag{1.4}$$

The second one (see Theorem 5) provides sufficient conditions to assure the existence of periodic solutions of the following differential system

$$x' = F(t, z, \varepsilon) = F_0(t, x) + \sum_{i=1}^{k} \varepsilon^i F_i(t, x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t, z) \in \mathbb{S}^1 \times \mathcal{D}. \tag{1.5}$$

Here $\mathbb{S}^1 = \mathbb{R}/T$, for some $T > 0$, and the assumption $t \in \mathbb{S}^1$ means that the system is T-periodic in the variable t. As usual $\delta_1(\varepsilon) = \mathcal{O}(\delta_2(\varepsilon))$ means that there exists a constant $c_0 > 0$, which does not depends on ε, such that $|\delta_1(\varepsilon)| \leq c_0 |\delta_2(\varepsilon)|$ for ε sufficiently small (see [68]).

The problem of existence of periodic solutions in system (1.5) can often be reduced to the problem of persistence of zeros of equation (1.4). Usually it is assumed that either

$g(z,0)$ vanishes in a submanifold of $\mathcal{Z} \subset \mathcal{D}$, or that the unperturbed differential system $x' = F_0(t,x)$ has a submanifold $\mathcal{Z} \subset \mathcal{D}$ of T-periodic solutions. In both cases $\dim(\mathcal{Z}) \leq n$.

We assume that for some $z^* \in \mathcal{Z}$, $g(z^*,0) = 0$. We shall study the persistence of this zero for the function (1.4), $g(x,\varepsilon)$, assuming that $|\varepsilon| \neq 0$ is sufficiently small. By *persistence* we mean the existence of continuous branches $\chi(\varepsilon)$ of simple zeros of $g(x,\varepsilon)$ (that is $g(\chi(\varepsilon),\varepsilon) = 0$) such that $\chi(0) = z^*$. It is well known that if the $n \times n$ matrix $\partial_x g(z^*,0)$ (the Jacobian matrix of the function g with respect to the variable x evaluated at $x = z^*$) is nonsingular then, from a direct consequence of the Implicit Function Theorem, there exists a unique smooth branch $\chi(\varepsilon)$ of zeros of $g(x,\varepsilon)$ such that $\chi(0) = z^*$. However if the matrix $\partial_x g(z^*,0)$ is singular (has non trivial kernel) we have to use the Lyapunov–Schmidt reduction method to find branches of zeros of g (see, for instance, [23]). Here we generalize some results from [8, 9, 51], providing a collection of functions f_i, $i = 1,\ldots,k$, each one called *bifurcation function of order* i, which control the persistence of zeros contained in $\mathcal{Z}$.

The problem of existence of periodic solutions of the differential system (1.5) goes back to the works of Malkin [56] and Roseau [67]. They have studied the case $k = 1$. Let $x(t,z,\varepsilon)$ denote the solution of system (1.5) such that $x(0,z,\varepsilon) = z$. In order to find initial conditions $z \in \mathcal{D}$ such that the solution $x(t,z,\varepsilon)$ is T-periodic we may consider the function $g(z,\varepsilon) = z - x(T,z,\varepsilon)$, and then try to use the results previously obtained about the persistence of zeros. Indeed, if $\mathcal{Z} \subset \mathcal{D}$ is a submanifold of T-periodic solutions of the unperturbed system $x' = F_0(t,x)$, then $g(z,0)$ vanishes on $\mathcal{Z}$. When $\dim(\mathcal{Z}) = n$ this problem is studied at an arbitrary order of ε, see [33, 53], even for nonsmooth systems. When $\dim(\mathcal{Z}) < n$, this approach has already been used in [8], up to order 1, and in [9, 10], up to order 2. In [51] this approach was used up to order 3 relaxing some hypotheses assumed in those previous 3 works. In [34] assuming the same hypotheses of [8, 9, 10] the authors studied this problem at an arbitrary order of ε. Here, following the ideas from [53, 51], we improve the results of [34] relaxing some hypotheses and developing the method in a more general way.

In summary, we use the Lyapunov–Schmidt reduction method for studying the zeros of functions like (1.4) when the Implicit Function Theorem cannot be directly applied. Another useful tool that we shall use to deal with this problem is the Browder degree theory (see Appendix B), which will allow to provide estimates for these zeros. Then we apply these previous results for studying the periodic solutions of differential systems like (1.5) through their bifurcation functions, provided by the higher order averaging theory.

The results are organized as follows. In Chapter 1 we present our main results on averaging theory. In Chapter 2 we provide the proofs of the main results. Then we start apply our results to study the periodic solutions of some relevant physical systems. In Chapter 3 we study the Maxwell-Bloch system and a 3D polynomial differential system. In Chapter 4 we study 17 differential systems, including the Fitzhugh-Nagumo system, the Noose-Hover system, the Wang-Chen system and the Wei system. In Chapter 5 we study the existence and stability of periodic solutions in the Lorenz differential system and the Thomas differential system. In Chapter 6 we study the periodic solutions and invariant tori in the generalized Van der Pol - Duffing differential system using Lyapunov coefficients and averaging theory. Finally, in Chapter 7 we study the periodic solutions in a hyperchaotic Lorenz differential system.

The results presented in Chapter 1, 2 and 3 were based on the preprint [18] and published papers [17] and [14]. The results presented in Chapter 4 are published in [13] and [15]. Chapter 5 contains results from [13] and [14]. The results in Chapter 6 are submited for publication. The results in Chapter 7 are published in [16].

1.1 Statements of the main results

Before we state our main results we need some preliminary concepts and definitions. Given p, q and L positive integers, $\gamma_j = (\gamma_{j1}, \dots, \gamma_{jp}) \in \mathbb{R}^p$ for $j = 1, \dots, L$ and $\overline{z} \in \mathbb{R}^p$. Let $G : \mathbb{R}^p \to \mathbb{R}^q$ be a sufficiently smooth function, then the L-th Frechet derivative of G at $\overline{z}$ is denoted by $\partial^L G(\overline{z})$, it is a symmetric L–multilinear map, which applied to a "product" of L p-dimensional vectors denoted as $\bigodot_{j=1}^L \gamma_j \in \mathbb{R}^{pL}$ gives

$$\partial^L G(\overline{z}) \bigodot_{j=1}^{L} \gamma_j = \left(\sum_{i_1,\dots,i_L=1}^{p} \frac{\partial^L G^1(\overline{z})}{\partial z_{i_1} \cdots \partial z_{i_L}} \gamma_{1i_1} \cdots \gamma_{Li_L}, \cdots, \sum_{i_1,\dots,i_L=1}^{p} \frac{\partial^L G^q(\overline{z})}{\partial z_{i_1} \cdots \partial z_{i_L}} \gamma_{1i_1} \cdots \gamma_{Li_L} \right). \tag{1.6}$$

The above expression is indeed the Gâteaux derivative

$$\begin{aligned} \partial^L G(\overline{z}) \bigodot_{j=1}^{L} \gamma_j &= \frac{\partial}{\partial \tau_1 \partial \tau_2 \dots \partial \tau_L} G\left(\overline{z} + \tau_1\gamma_1 + \tau_2\gamma_2 + \cdots + \tau_L\gamma_L\right) \Big|_{\tau_1=\cdots=\tau_L=0} \\ &= \partial\Big(\dots \partial\big(\partial G(\overline{z})\gamma_1\big)\gamma_2 \dots \Big)\gamma_L. \end{aligned}$$

We take ∂^0 as the identity operator.

1.1.1 The Lyapunov–Schmidt reduction method

We consider the function

$$g(z,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}), \tag{1.7}$$

where $g_i : \mathcal{D} \to \mathbb{R}^n$ is a $\mathcal{C}^{k+1}$ function, $k \geq 1$, for $i = 0, 1, \dots, k$, being $\mathcal{D}$ an open bounded subset of $\mathbb{R}^n$. For $m < n$, let V be an open bounded subset of $\mathbb{R}^m$ and $\beta : \mathrm{Cl}(V) \to \mathbb{R}^{n-m}$ a $\mathcal{C}^{k+1}$ function, such that

$$\mathcal{Z} = \{ z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \mathrm{Cl}(V) \} \subset \mathcal{D}. \tag{1.8}$$

As usual $\mathrm{Cl}(V)$ denotes the closure of the set V.

As the main hypothesis we assume that

(H_a) the function g_0 vanishes on the m–dimensional submanifold $\mathcal{Z}$ of $\mathcal{D}$.

Using the Lyapunov–Schmidt reduction method we shall develop the bifurcation functions of order i, for $i = 1, 2, \dots, k$, which control, for $|\varepsilon| \neq 0$ small enough, the existence of branches of zeros $z(\varepsilon)$ of (1.7) bifurcating from $\mathcal{Z}$, that is from $z(0) \in \mathcal{Z}$. With this purpose we introduce some notation. The functions $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and

$\pi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the projections onto the first m coordinates and onto the last $n-m$ coordinates, respectively. For a point $z \in \mathcal{D}$ we also consider $z = (a,b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. We define $\partial_b^L \pi g_{i-l}(z_\alpha)$ by following the notation (1.6), taking $p = n-m$, $q = m$, $\overline{z} = \beta(\alpha)$ and $G : b \to \pi g_{i-l}(\alpha, b)$. Let S_l be the set of all l-tuples of non–negative integers $(c_1, c_2, \cdots, c_l)$ satisfying $c_1+2c_2+\cdots+lc_l = l$, $L = c_1+c_2+\cdots+c_l$, and S_i' is the set of all $(i-1)$-tuples of non–negative integers satisfying $c_1+2c_2+\cdots+(i-1)c_{i-1} = i$, $I' = c_1 + c_2 + \cdots + c_{i-1}$. From (1.6) we define

$$\begin{aligned}\partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} = \Bigg(& \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi g_{i-l}^1(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l}, \cdots , \\ & \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi g_{i-l}^m(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l} \Bigg) \Bigg|_{(a,b)=z_\alpha}\end{aligned}$$

and

$$\begin{aligned}\partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} = \Bigg(& \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi^\perp g_{i-l}^{m+1}(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l}, \cdots , \\ & \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi^\perp g_{i-l}^n(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l} \Bigg) \Bigg|_{(a,b)=z_\alpha}.\end{aligned}$$

For $i = 1, 2, \ldots, k$ we define the *bifurcation functions* $f_i : \mathrm{Cl}(V) \to \mathbb{R}^m$ of order i as

$$f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}, \quad \text{and} \tag{1.9}$$

$$\mathcal{F}^k(\alpha, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i f_i(\alpha),$$

where $\gamma_i : V \to \mathbb{R}^{n-m}$, for $i = 1, 2, \ldots, k$, are defined recurrently as

$$\begin{aligned}\gamma_1(\alpha) &= -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) &= -i! \Delta_\alpha^{-1} \Bigg(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{I'} \pi^\perp g_0(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \\ &\qquad + \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} \Bigg).\end{aligned} \tag{1.10}$$

with $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$.

We clarify that $S_0 = S_0' = \emptyset$, and when $c_j = 0$, for some j, then the term γ_j does not appear in the "product" $\bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}$.

The next theorem is the first main result of this chapter. For sake of simplicity, we take $f_0 = 0$.

Theorem 2. *Let Δ_α denote the lower right corner of the $(n-m)\times(n-m)$ matrix of the Jacobian matrix $D\,g_0(z_\alpha)$. In additional to hypothesis (H_a) we assume that*

(i) *for each $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$;*

(ii) *for some $r \in \{1,\dots,k\}$, $f_1 = f_2 = \cdots = f_{r-1} = 0$ and f_r is not identically zero;*

(iii) *there exists a small parameter $\varepsilon_0 > 0$ such that for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ there exists $a_\varepsilon \in V$ satisfying $\mathcal{F}^k(a_\varepsilon, \varepsilon) = 0$; and*

(iv) *there exist a constant $P_0 > 0$ and a positive integer $l \leq (k+r+1)/2$ such that*

$$\left|\partial_\alpha \mathcal{F}^k(a_\varepsilon,\varepsilon)\cdot \alpha\right| \geq P_0|\varepsilon|^l|\alpha|, \quad \textit{for} \quad \alpha \in V.$$

Then, for $|\varepsilon| \neq 0$ sufficiently small, there exists $z(\varepsilon)$ such that $g(z(\varepsilon),\varepsilon) = 0$ with $|\pi^\perp z(\varepsilon) - \pi^\perp z_{a_\varepsilon}| = \mathcal{O}(\varepsilon)$ and $|\pi\, z(\varepsilon) - \pi\, z_{a_\varepsilon}| = \mathcal{O}(\varepsilon^{k+1-l})$.

In the next corollary we present a classical result in the literature, which is a direct consequence of Theorem 2.

Corollary 3. *In addiction to hypothesis (H_a), assume that $f_1 = f_2 = \cdots = f_{k-1} = 0$ and that for each $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$. If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$ and $\det\left(Df_k(\alpha^*)\right) \neq 0$, then there exists a branch of zeros $z(\varepsilon)$ with $g(z(\varepsilon),\varepsilon) = 0$ and $|z(\varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

Theorem 2 and Corollary 3 are proved in Section 2.1.

1.1.2 Continuation of periodic solutions

We consider the following $\mathcal{C}^{k+1}$ differential system

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t,z,\varepsilon) \in \mathbb{S}^1 \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0). \tag{1.11}$$

Here $\mathcal{D} \subset \mathbb{R}^n$ is an open and bounded set, $\varepsilon_0 > 0$, and the prime denotes derivative with respect to the time t. We denote the right–hand side of equation (1.11) by $F(t,x,\varepsilon)$. We say that the differential system (1.11) is in the *normal form for applying the averaging theory*. Given $z \in \mathcal{D}$ we denote by $x(t,z,\varepsilon)$ the solution of the differential system (1.11) such that $x(0,z,\varepsilon) = z$. As our basic hypothesis we assume that:

(H) There exists a manifold $\mathcal{W} \subset \mathcal{D}$ such that, for each $z \in \mathcal{W}$, the solution $x(t,z,0)$ of the unperturbed system is T-periodic.

Thus we have the following result.

Lemma 4 (Fundamental Lemma)**.** *Let $x(t,z,\varepsilon)$ be the solution of the $\mathcal{C}^{k+1}$ T-periodic differential system* (1.11) *such that $x(0,z,\varepsilon) = z$. Then the equality*

$$x(t,z,\varepsilon) = x(t,z,0) + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t,z)}{i!} + \mathcal{O}(\varepsilon^{k+1}) \tag{1.12}$$

holds for $(t,z) \in \mathbb{S}^1 \times \mathcal{D}$. *Here the functions* y_i *for* $1 \le i \le k$, *are given recursively as*

$$
\begin{aligned}
y_1(t,z) =& Y(t,z) \int_0^t Y(s,z)^{-1} F_1(s, x(s,z,0)) ds,\\
y_i(t,z) =& i!\, Y(t,z) \int_0^t Y(s,z)^{-1} \Bigg(F_i(s, x(s,z,0))\\
&+ \sum_{S_i'} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_{i-1}!(i-1)!^{b_{i-1}}} \partial^{I'} F_0(s, x(s,z,0)) \bigodot_{j=1}^{i-1} y_j(s,z)^{b_j}\\
&+ \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_{i-l}(s, x(s,z,0)) \bigodot_{j=1}^{l} y_j(s,z)^{b_j} \Bigg) ds,
\end{aligned}
$$

where $Y(t,z)$ *is a fundamental matrix solution of the linear differential system* $y' = \partial_x F_0(t, x(t,z,0)) y$, *being* $\partial_x F_0(t,x)$ *the Jacobian matrix of the function* $F_0(t,x)$.

From hypothesis (H) we see that there exists an open set $U_1 \subset \mathcal{D}$ and $\varepsilon_1 > 0$ such that, for each $z \in \overline{U}_1$ and $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$, the solution $x(t,z,\varepsilon)$ is defined on the interval $[0, t_{(z,\varepsilon)})$, with $t_{(z,\varepsilon)} > T$.

A *displacement function* $d : U_1 \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^n$ can be defined as $d(z,\varepsilon) = x(T,z,\varepsilon) - z$. Notice that a solution (z^*, ε^*) of the equation $d(z,\varepsilon) = 0$ corresponds to a T-periodic solution of the differential system (1.11) with $\varepsilon = \varepsilon^*$ and initial condition z^*. From (1.12), the displacement function reads

$$
d(z,\varepsilon) = x(T,z,0) - z + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(T,z)}{i!} + \mathcal{O}(\varepsilon^{k+1}). \tag{1.13}
$$

The equation $d(z,\varepsilon) = 0$ is equivalent to

$$
g(z,\varepsilon) \stackrel{\text{def}}{=} Y(T,z)^{-1} d(z,\varepsilon) = 0, \tag{1.14}
$$

and from (1.13) equation (1.14) writes

$$
g(z,\varepsilon) = g_0(z) + \sum_{i=1}^{k} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}),
$$

where $g_0(z) = Y(T,z)^{-1} \left(x(T,z,0) - z \right)$ and

$$
g_i(z) = Y(T,z)^{-1} \frac{y_i(T,z)}{i!}, \; i = 1, 2 \ldots, k, \tag{1.15}
$$

are usually called the *averaged function* of order i. By abuse of notation, the function g_0 is called the averaged function of order 0. Notice that $g_0(z) = 0$ if, and only if, the solution $x(t,z,0)$ of the unperturbed system is T–periodic. Therefore, from hypothesis (H), $g_0(z) = 0$ for every $z \in \mathcal{Z}$.

The averaging theory for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions of system (1.11) by studying the solutions of equation (1.14).

In [17] it was assumed that $g_0 \not\equiv 0$. Here we assume that $g_s \not\equiv 0$ is the first nonvanishing averaged function, where $0 \le s < k$. As our main hypotheses we assume that

($\mathcal{H}$) Let $g_s \not\equiv 0$, for $0 \leq s < k$, be the first nonvanishing averaged function. Assume that there exist $m < n$, V an open bounded subset of $\mathbb{R}^m$, and a $\mathcal{C}^{k+1}$ function $\beta : \overline{V} \to \mathbb{R}^{n-m}$ such that $\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\} \subset \mathcal{D}$, and $g_s(z_\alpha) = 0$ for every $\alpha \in \overline{V}$.

Notice that ($\mathcal{H}$) implies (H). Indeed, if $s = 0$, then (H) holds by taking $\mathcal{Z} = \mathcal{W}$. Otherwise ($H$) holds by taking $\mathcal{Z} = \mathcal{D}$.

From hypothesis ($\mathcal{H}$) and Lemma 4 equation (1.13) is equivalente to

$$h(z,\varepsilon) \stackrel{\text{def}}{=} \frac{g(z,\varepsilon)}{\varepsilon^s} = g_s(z) + \sum_{i=1}^{k-s} \varepsilon^i g_{s+i}(z) + \mathcal{O}(\varepsilon^{k-s+1}) = 0. \tag{1.16}$$

From Theorem 2 the bifurcation functions corresponding to equation (1.16) are

$$f_i(\alpha) = \pi g_{s+i}(z_\alpha) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi g_{s+i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}, \tag{1.17}$$

$$\mathcal{F}^{k-s}(\alpha,\varepsilon) = \sum_{i=1}^{k-s} \varepsilon^i f_i(\alpha), \tag{1.18}$$

where $\gamma_i : V \to \mathbb{R}^{n-m}$, for $i = 1, 2, \ldots, k-s$, are defined recurrently as

$$\begin{aligned}
\gamma_1(\alpha) =& -\Delta_\alpha^{-1} \pi^\perp g_{s+1}(z_\alpha) \quad \text{and} \\
\gamma_i(\alpha) =& -i! \Delta_\alpha^{-1} \Bigg(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{I'} \pi^\perp g_s(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \\
&+ \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{s+i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} \Bigg),
\end{aligned}$$

with $\Delta_\alpha = \dfrac{\partial \pi^\perp g_s}{\partial b}(z_\alpha)$.

In what follows we shall state a slightly improvement of Theorem B from [17], which is suitable to a wider range of applications.

Theorem 5. *Assume hypothesis ($\mathcal{H}$) holds. Consider the Jacobian matrix*

$$\partial g_s(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix},$$

where $\Lambda_\alpha = \partial_a \pi g_s(z_\alpha)$, $\Gamma_\alpha = \partial_b \pi g_s(z_\alpha)$, $B_\alpha = \partial_a \pi^\perp g_s(z_\alpha)$ and $\Delta_\alpha = \partial_b \pi^\perp g_s(z_\alpha)$. In additional to hypothesis ($\mathcal{H}$) we suppose that

(*i*) *for each $\alpha \in \overline{V}$, $\det(\Delta_\alpha) \neq 0$;*

(*ii*) *for some $r \in \{0, \ldots, k-s\}$, $f_1 = f_2 = \cdots = f_{r-1} = 0$ and f_r is not identically zero;*

(*iii*) *there exists $\overline{\varepsilon} > 0$ such that for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$ there exists $a_\varepsilon \in V$ satisfying $\mathcal{F}^{k-s}(a_\varepsilon, \varepsilon) = 0$; and* (*iv*) *there exist a constant* $P_0 > 0$ *and a positive integer* $l \leq (k - s + r + 1)/2$ *such that*

$$\left|\partial_\alpha \mathcal{F}^{k-s}(a_\varepsilon, \varepsilon) \cdot \alpha\right| \geq P_0 |\varepsilon^l| |\alpha|, \quad \textit{for} \quad \alpha \in V.$$

Then for $|\varepsilon| \neq 0$ *sufficiently small there exists a* T*-periodic solution* $\varphi(t, \varepsilon)$ *of system* (1.11) *such that* $|\pi\, \varphi(0, \varepsilon) - \pi\, z_{a_\varepsilon}| = \mathcal{O}(\varepsilon^{k-s+1-l})$, *and* $|\pi^\perp \varphi(0, \varepsilon) - \pi^\perp z_{a_\varepsilon}| = \mathcal{O}(\varepsilon)$.

In the next corollary we present a classical result in the literature, which is a direct consequence of Corollary 3.

Corollary 6. *In addiction to hypothesis* $(\mathcal{H})$ *we assume that* $f_1 = f_2 = \cdots = f_{r-1} = 0$, $r = k - s$ *and that for each* $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$. *If there exists* $\alpha^* \in V$ *such that* $f_r(\alpha^*) = 0$ *and* $\det\left(Df_r(\alpha^*)\right) \neq 0$, *then there exists a* T*-periodic solution* $\varphi(t, \varepsilon)$ *of* (1.11) *such that* $|\varphi(0, \varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.

Lemma 4, Theorem 5 and Corollary 6 are proved in Section 2.2.

It is worth to emphasize that Theorem 5 is still true when $m = n$. In fact, assuming that V is an open subset of $\mathbb{R}^n$ then $\mathcal{Z} = \mathrm{Cl}(V) \subset \mathcal{D}$ and the projections π and $\pi^\perp$ become the identity and the null operator respectively. Moreover, in this case the bifurcation functions $f_i : V \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k$, are the averaged functions $f_i(\alpha) = g_i(\alpha)$ defined in (1.15). Consider $m = n$, $z_\alpha = \alpha \in \mathcal{Z}$ and the hypothesis $(\mathcal{H})$. Thus the result of Theorem 5 holds without any assumption about Δ_α. Thus we have the following corollary, which recover the main result from [53].

Corollary 7. *Assume that* $g_s \equiv 0$. *If there exists* $z^* \in \Omega$ *such that* $g_{s+1}(z^*) = 0$ *and* $Dg_{s+1}(z^*) \neq 0$, *then there exists a* T*-periodic solution* $x(t, z(\varepsilon), \varepsilon)$ *for system* (1.11) *such that* $z(0) = z^*$.

Now we use functions $\alpha(\varepsilon)$, γ_i and f_i to study the stability of the periodic solution $\varphi(t, \varepsilon)$.

1.2 Stability of the periodic solutions

A fundamental notion in qualitative theory of differential equations is the hyperbolicity. Here a constant matrix will be called *hyperbolic* if its eigenvalues lie out of the unitary circle of the complex plane, in which case its *index* is the number of eigenvalues outside the unitary circle.

Consider a matrix function $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + \varepsilon^k A_k$ depending on a parameter ε. If A_0 is hyperbolic of index i, then one can see that for $\varepsilon > 0$ sufficiently small $A(\varepsilon)$ will be hyperbolic with the same index i.

If A_0 is not hyperbolic the placement of the eigenvalues of $A(\varepsilon)$ may be hard to determine. To deal with this problem we use a method introduced by Murdock and Robinson in [62, 61]. The matrix $A(\varepsilon)$ is called *k-hyperbolic of index i* if for every smooth matrix function $B(\varepsilon)$ there exists an $\varepsilon_0 > 0$ such that $A(\varepsilon) + \varepsilon^k B(\varepsilon)$ is hyperbolic of index i for all ε in the interval $0 < \varepsilon < \varepsilon_0$.

The stability properties of the periodic solution $\varphi(t, \varepsilon)$ will be provided using the k-determined hyperbolicity method, as it was presented in [60, Chapter 3].

For $|\varepsilon| \neq 0$ sufficiently small let $\varphi(t,\varepsilon) = x(t, z(\varepsilon), \varepsilon)$ be a $T-$periodic solution of the differential system (1.11) given by Theorem 5 such that $z(0) = z_{\alpha^*} \in \mathcal{Z}$. The *Poincaré Map* related to $\varphi(t,\varepsilon)$ is given by

$$\Pi(z,\varepsilon) \stackrel{\text{def}}{=} x(T, z, \varepsilon) = z + d(z,\varepsilon). \tag{1.19}$$

Clearly $z(\varepsilon)$ is a fixed point of $\Pi(\cdot,\varepsilon)$. It is well known that the stability of the fixed point $z(\varepsilon)$ of the Poincaré map $\Pi(\cdot,\varepsilon)$ yields the stability of the T-periodic solution $\varphi(t,\varepsilon)$. More specifically, if the norm of each eigenvalue of $\partial_z\Pi(z(\varepsilon),\varepsilon)$ is less than 1, then the periodic solution $\varphi(t,\varepsilon)$ is stable. On the other hand, if there exists an eigenvalue of $\partial_z\Pi(z(\varepsilon),\varepsilon)$ with norm greater than 1, then the periodic solution $\varphi(t,\varepsilon)$ is unstable. From (1.19), our goal in is to show how the power series of $z(\varepsilon)$ around $\varepsilon = 0$ can be used to provide the stability of the $T-$periodic solutions $x(t, z(\varepsilon), \varepsilon)$ provided in Theorem 5. As these solutions are essentially non-hyperbolic, due to existence of a continuum of zeros of the first coefficient function of (1.16), the question about its stability can be reduced to the study of the k-determined hyperbolicity of the Jacobian matrix $\partial_z d(z(\varepsilon),\varepsilon)$.

For the sake of further applications the first result of this section is to write the formal power series of the initial condition $z(\varepsilon) = \varphi(0,\varepsilon)$ around $\varepsilon = 0$, where $\varphi(t,\varepsilon)$ is the $T-$periodic solution provided in Theorem 5.

The next result reveals how the higher order averaged functions can be used for determining the stability of the periodic solution $\mathbf{x}(t, z(\varepsilon), \varepsilon)$.

Lemma 8. *Let a_ε be the one given in hypothesis (iii) of Theorem 5 and let $x(t, z(\varepsilon), \varepsilon) = \varphi(t,\varepsilon)$ be the periodic solution of the differential system* (1.11) *provided in Theorem 5. If*

$$a_\varepsilon = \alpha_0 + \varepsilon\alpha_1 + \cdots + \varepsilon^{k-s-l}\alpha_{k-s-l} + \mathcal{O}(\varepsilon^{k-s-l+1}), \tag{1.20}$$

with $\alpha_i \in \mathbb{R}^m$ for all $0 \le i \le k-s-l$. Then we can write initial condition of the periodic orbit as

$$z(\varepsilon) = \sum_{i=0}^{k-s-l} \varepsilon^i\big(\alpha_i, \beta_i\big) + \mathcal{O}\big(\varepsilon^{k-s-l+1}\big), \tag{1.21}$$

where $\beta_0 = \beta(\alpha_0)$ and for all $1 \le i \le k-s-l$,

$$\beta_i = \gamma_i(\alpha_0) + \sum_{j=1}^{i}\sum_{S_j} \frac{1}{c_1!\, c_2!2!^{c_2}\cdots c_j!j!^{c_j}}\gamma_{i-j}^{(J)}(\alpha_0)\bigodot_{s=1}^{j}(s!\alpha_s)^{c_s}. \tag{1.22}$$

The next result provides the Taylor expansion at $\varepsilon = 0$ of the Jacobian matrix of the displacement function (1.13) evaluated at $z(\varepsilon) = \varphi(0,\varepsilon)$, where $\varphi(t,\varepsilon)$ is the T-periodic function provided in Theorem 5.

Lemma 9. *We assume that system* (1.11) *satisfies the hypotheses of Theorem 5 having the T-periodic solution $\varphi(t,\varepsilon)$. Moreover, let $z(\varepsilon) = \varphi(0,\varepsilon)$ and a_ε from statement (iii) of Theorem 5 written in the form* (1.20). *Thus the Jacobian matrix of displacement map* (1.13) *at $z = z(\varepsilon)$ can be written as*

$$\partial_z d(z(\varepsilon),\varepsilon) = \varepsilon^s A(\varepsilon) + \mathcal{O}(\varepsilon^{k-l+1}),$$

where $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + \varepsilon^{k-s-l} A_{k-s-l}$ *where* A_j *is an* $n \times n$ *constant matrix for all* $0 \leq j \leq k-s-l$. *More precisely, we have* $A_0 = \partial y_s(T, z_0)$ *and*

$$A_j = \sum_{i=0}^{j} \frac{1}{(j-i)!} \sum_{S_i} \frac{1}{b_1! \cdots b_i!(i-1)!^{b_i}} \partial_z^{I+1} y_{s+j-i}(T, z_0) \bigodot_{u=1}^{i} (u! z_u)^{b_u},$$

for $1 \leq j \leq k-s-l$, *with* $z_i = (\alpha_i, \beta_i)$ *given in* (1.21) *and* l *as in Theorem 5.*

Consequently the Jacobian matrix of the Poincaré map becomes

$$D\Pi(z, \varepsilon) \stackrel{\text{def}}{=} M(\varepsilon) + \mathcal{O}(\varepsilon^{k-l+1}), \tag{1.23}$$

with $M(\varepsilon) = Id + \varepsilon^s A(\varepsilon)$. Now we can present our result on the stability of the non-hyperbolic T-periodic solution $x(t, z(\varepsilon), \varepsilon)$ provided in Theorem 5.

Theorem 10. *We assume that system* (1.11) *has a* T*-periodic solution* $x(t, z(\varepsilon), \varepsilon)$ *as stated in Lemma* 9, *and that the Jacobian matrix of the Poincaré map at* $z(\varepsilon)$ *has the form* (1.23) *with* $M(\varepsilon)$ *hyperbolic for* $|\varepsilon|$ *sufficiently small. If there exists a matrix* $T(\varepsilon)$ *such that* $T(\varepsilon)^{-1} M(\varepsilon) T(\varepsilon) = \Lambda(\varepsilon)$, *where*

$$\Lambda(\varepsilon) = \begin{bmatrix} \lambda_1(\varepsilon) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n(\varepsilon) \end{bmatrix} = \varepsilon^{r_1} \Lambda_1 + \cdots + \varepsilon^{r_j} \Lambda_j;$$

with $r_1 < r_2 < \cdots < r_j < R = k - l + 1$ *rational numbers, and* $\Lambda_1, \ldots, \Lambda_j$ *diagonal matrices. Then there exists an* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the eigenvalues of the Jacobian matrix* $D\Pi(z, \varepsilon)$ *are approximately equal to* $\lambda_i(\varepsilon)$ *with error* $\mathcal{O}(\varepsilon^R)$. *Consequently the matrices* $M(\varepsilon)$ *and* $D\Pi(z, \varepsilon)$ *have the same hyperbolicity type.*

The result of Theorem 10 is strongly related with the Theorem 3.7.7 of [60]. Obtaining the matrix $T(\varepsilon)$ may be the main difficulty of applying Theorem 10. In some cases it may be necessary a sequence of linear transformations and normalization in order to obtain $T(\varepsilon)$, see [60, Section 3.7]. This task always comes down to the solution of a homological equation such as

$$\mathcal{L} U_j = K_j - B_j,$$

where

$$\mathcal{L} = \mathbb{L}_Y : gl(n) \to gl(n),$$

K_j is known at the jth stage of the calculation, and B_j and U_j are to be determined and $\mathbb{L}_Y$ is the Lie operator, i.e. $\mathbb{L}_Y X = [X, Y] = XY - YX$. In this work we shall use Theorem 5 to study the Hopf or the zero–Hopf bifurcation in some three dimensional systems. Moreover Corollary 19 in Appendix 2.7 provides sufficient conditions for the existence of the matrix $T(\varepsilon)$. This will allow to use Theorem 10 for studying the stability of the bifurcated periodic orbits detected by Theorem 5.

Finally we shall show that the hypotheses of Lemma 8 are not very restrictive. We shall provide the expressions of the $\alpha_i' s$ in Lemma 8 in terms of the bifurcation functions (1.17).

Proposition 11. *Assume that $0 \leq r < k$ is the first subindex such that $f_r(\alpha) \not\equiv 0$ as given by hypothesis (ii) of Theorem 5. If there exist $\alpha^* \in V \subset \mathbb{R}^m$ such that $f_r(\alpha^*) = 0$ and $\det(\partial f_r(\alpha^*)) \neq 0$. Then there exists a unique $a_\varepsilon \in V$ such that:*

(a) $a_\varepsilon = \alpha_0 + \varepsilon\alpha_1 + \cdots + \varepsilon^k\alpha_k + \mathcal{O}(\varepsilon^{k+1})$ with $\alpha_i \in \mathbb{R}^n$ for all $1 \leq i \leq k$ satisfying $\mathcal{F}^k(a_\varepsilon, \varepsilon) = 0$, and

(b) where the coefficients are $\alpha_0 = \alpha^*$, $\alpha_1 = -Df_r(\alpha^*)^{-1} f_{r+1}(\alpha^*)$ and for $2 \leq i \leq k-1$

$$\begin{aligned}\alpha_i =& \frac{-Df_r}{i!}(\alpha^*)^{-1}\Bigg(\sum_{S'_i} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}!(i-1)!^{c_{i-1}}} f_r^{(I')}(\alpha^*) \bigodot_{j=1}^{i-1} \alpha^{(j)}(0)^{c_j} \\ &+ \sum_{l=0}^{i-1}\sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} f_{i-l+r}^{(L)}(\alpha^*) \bigodot_{j=1}^{l} \alpha^{(j)}(0)^{c_j}\Bigg),\end{aligned}$$

Proposition 11 is particularly useful to study the stability of the periodic orbits detected by Corollary 6. This result will be applied several times in this work. Thus we present now a reformulation of Corollary 6 and Theorem 10 that will be used in the applications presented in the next chapters.

Theorem 12. *Let $s \in \mathbb{R}$ such that s is the first subindex such that $g_s \not\equiv 0$. In addition to hypothesis $(\mathcal{H})$ assume that*

(i) the averaged function g_s vanishes on the manifold (1.8). That is $g_s(z_\alpha) = 0$ for all $\alpha \in \overline{V}$, and

(ii) the Jacobian matrix

$$Dg_s(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix},$$

where $\Lambda_\alpha = D_a\pi g_s(z_\alpha)$, $\Gamma_\alpha = D_b\pi g_s(z_\alpha)$, $B_\alpha = D_a\pi^\perp g_s(z_\alpha)$ and $\Delta_\alpha = D_b\pi^\perp g_s(z_\alpha)$, satisfies that $\det(\Delta_\alpha) \neq 0$ for all $\alpha \in \overline{V}$.

We define the functions

$$\begin{aligned}f_1(\alpha) =& -\Gamma_\alpha\Delta_\alpha^{-1}\pi^\perp g_{s+1}(z_\alpha) + \pi g_{s+1}(z_\alpha),\\ f_2(\alpha) =& \frac{1}{2}\Gamma_\alpha\gamma_2(\alpha) + \frac{1}{2}\frac{\partial^2 \pi g_s}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi g_{s+1}}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi g_{s+2}(z_\alpha),\\ \gamma_1(\alpha) =& -\Delta_\alpha^{-1}\pi^\perp g_{s+1}(z_\alpha),\\ \gamma_2(\alpha) =& -\Delta_\alpha^{-1}\left(\frac{\partial^2\pi^\perp g_s}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial\pi^\perp g_{s+1}}{\partial b}(z_\alpha)\gamma_1(\alpha) + 2\pi^\perp g_{s+2}(\alpha)\right).\end{aligned} \tag{1.24}$$

Then the following statements hold.

(a) If there exists $\alpha^* \in V$ such that $f_1(\alpha^*) = 0$ and $\det(Df_1(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small there is an initial condition $z(\varepsilon) \in U$ such that $z(0) = z_{\alpha^*}$ and the solution $x(t, z(\varepsilon), \varepsilon)$ of system (1.11) is T-periodic.

(*b*) *Assume that* $f_1 \equiv 0$. *If there exists* $\alpha^* \in V$ *such that* $f_2(\alpha^*) = 0$ *and* $\det(Df_2(\alpha^*)) \neq 0$, *for* $|\varepsilon| \neq 0$ *sufficiently small there is an initial condition* $z(\varepsilon) \in U$ *such that* $z(0) = z_{\alpha^*}$ *and the solution* $x(t, z(\varepsilon), \varepsilon)$ *of system* (1.11) *is* T*-periodic.*

The next result provides the stability type of the periodic solutions detected by Theorem 12(a). Here *diagonalizable* means that the matrix has n distinct eigenvalues.

Theorem 13. *Consider* s, Γ_α, Δ_α, f_1 *and* f_2 *as defined in Theorem 12 and the Jacobian matrices* $Dy_s(T, z) = (p_{ij}(z))$ *and* $Dy_{s+1}(T, z) = (q_{ij}(z))$. *Assume that there exists* $\alpha^* \in V$ *such that* $f_1(\alpha^*) = 0$ *and* $\det(Df_1(\alpha^*)) \neq 0$. *We define the matrix function*

$$A(\varepsilon) = A_0 + \varepsilon A_1, \tag{1.25}$$

where

$$\begin{aligned} A_0 =& Dy_s(T, z_{\alpha^*}), && (1.26)\\ A_1 =& \left(Dp_{ij}(z_{\alpha^*}).z_1 + q_{ij}(z_{\alpha^*})\right), && (1.27)\\ z_1 =& \left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*),\, D\beta(\alpha^*)\left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*)\right) + \gamma_1(\alpha^*)\right). && (1.28) \end{aligned}$$

We assume that $A(\varepsilon)$ *satisfies the following statements:*

(s_1) A_0 *is diagonalizable and* $s > 0$, *or* $Id + A_0$ *is diagonalizable and* $s = 0$; *and*

(s_2) $Id + \varepsilon^s A_0 + \varepsilon^{s+1} A_1$ *is hyperbolic for all* ε *sufficiently small.*

Thus the Poincaré map of the periodic solution $x(t, z(\varepsilon), \varepsilon)$ *is* $s + 2-$*hyperbolic.*

In other words this last result says that the hyperbolicity of the $x(t, z(\varepsilon), \varepsilon)$ can be investigated using the $\lambda_i(\varepsilon) + \mathcal{O}(\varepsilon^{s+2})$, where $\lambda_{i's}(\varepsilon)$ are the eigenvalues of $Id + \varepsilon^s A_0 + \varepsilon^{s+1} A_1$. In the next chapter we prove the results here presented.