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On a family of rational perturbations of the doubling map

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The goal of this paper is to investigate the parameter plane of a rational family of perturbations of the doubling map given by the Blaschke products $B_a(z) = z^3(z-a)/(1-\bar{a}z)$. First we study the basic properties of these maps such as the connectivity of the Julia set as a function of the parameter a. We use techniques of quasiconformal surgery to explore the relation between certain members of the family and the degree 4 polynomials $(\overline{\bar{z}^2+c})^2+c$. In parameter space, we classify the different hyperbolic components according to the critical orbits and we show how to parametrize those of disjoint type.

Keywords: holomorphic dynamics; Blaschke products; circle maps; polynomial-like mappings

1. Introduction

Given a rational map $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ denotes the Riemann sphere, we consider the dynamical system given by the iterates of f. The Riemann sphere splits into two totally f-invariant subsets: the *Fatou set* $\mathcal{F}(f)$, which is defined to be the set of points $z\in\hat{\mathbb{C}}$ where the family $\{f^n, n\in\mathbb{N}\}$ is normal in some neighbourhood of z, and its complement, the *Julia set* $\mathcal{J}(f)$. The dynamics of the points in $\mathcal{F}(f)$ are stable whereas the dynamics in $\mathcal{J}(f)$ present chaotic behaviour. The Fatou set $\mathcal{F}(f)$ is open and therefore $\mathcal{J}(f)$ is closed. Moreover, if the degree of the rational map f is greater than or equal to 2, then the Julia set $\mathcal{J}(f)$ is not empty and is the closure of the set of repelling periodic points of f.

The connected components of $\mathcal{F}(f)$, called *Fatou components*, are mapped under f among themselves. Sullivan [33] proved that any Fatou component of a rational map is either periodic or preperiodic. By means of the Classification Theorem (see, e.g. [20]), any periodic Fatou component of a rational map is either the basin of attraction of an attracting or parabolic cycle or is a simply connected rotation domain (a Siegel disc) or is a doubly connected rotation domain (a Herman ring). Moreover, any such component is somehow related to a *critical point*, i.e. a point $z\in\hat{\mathbb{C}}$ such that $f'(z)=0$. Indeed, the basin of attraction of an attracting or parabolic cycle contains, at least, a critical point whereas Siegel discs and Herman rings have critical orbits accumulating on their boundaries. For a background on the dynamics of rational maps we refer to [3,20].

The aim of this paper is to study the dynamics of the degree 4 Blaschke products given by

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