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New results on averaging theory and applications

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Abstract. The usual averaging theory reduces the computation of some periodic solutions of a system of ordinary differential equations, to find the simple zeros of an associated averaged function. When one of these zeros is not simple, i.e., the Jacobian of the averaged function in it is zero, the classical averaging theory does not provide information about the periodic solution associated to a non-simple zero. Here we provide sufficient conditions in order that the averaging theory can be applied also to non-simple zeros for studying their associated periodic solutions. Additionally, we do two applications of this new result for studying the zero–Hopf bifurcation in the Lorenz system and in the Fitzhugh–Nagumo system.

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1. Introduction and statement of the main result

In this paper we introduce an improvement to the first-order averaging theorem and use it to study the zero–Hopf bifurcations of periodic orbits which take place in the Lorenz and Fitzhugh–Nagumo systems.

The first-order averaging theorem, as presented in [17], can be used to determine periodic orbits coming from the simple zeros of the averaged function, see Theorem 1. Here we use the Lyapunov–Schmidt reduction method (see Lemma 7) in order to make the averaging theorem able to determine periodic solutions coming from degenerated zeros of the averaged function, i.e., zeros for which the Jacobian determinant of the averaged function vanishes, see Theorem 2.

Here a *zero–Hopf equilibrium* is an equilibrium point of a three-dimensional autonomous differential system, which has a zero and a pair of pure imaginary eigenvalues. In general, the *zero–Hopf bifurcation* is a two-parameter unfolding of a three-dimensional autonomous differential equation with a zero–Hopf equilibrium. This kind of zero–Hopf bifurcations has been studied by Guckenheimer, Han, Holmes, Kuznetsov in [8–10,12], and it was shown that some complicated invariant sets can bifurcate from the isolated zero–Hopf equilibrium doing the unfolding. Due to the lack of a general theory describing all these kinds of bifurcations that the unfolding of a zero–Hopf bifurcation can produce, most of the systems exhibiting this kind of bifurcation must be studied directly.

Using Theorem 2, here obtained, we could detect the bifurcation of a periodic orbit from a zero–Hopf bifurcation in the famous Lorenz system of differential equations. As far as we know this was the first time this periodic solution was detected. We also apply this theorem to detect the bifurcation of new periodic solutions in the Fitzhugh–Nagumo system, improving the results obtained in [4].

1.1. Averaging theory

The averaging method is a classical theory for studying nonlinear dynamical systems and their periodic solutions. It was conceived by Lagrange in the eighteenth century, without formal proof, what was only

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