STABILITY OF PERIODIC ORBITS IN THE AVERAGING THEORY: APPLICATIONS TO LORENZ AND THOMAS' DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the kind of stability of the periodic orbits provided by higher order averaging theory. We apply these results for determining the $k-$hyperbolicity of some periodic orbits of the Lorenz and Thoma's differential system.

1. Introduction and statement of our main result

The averaging theory is a classical method for studying the solutions of the non-linear dynamical systems, and in particular their periodic solutions. For a general introduction to the averaging theory see the book of Sanders, Verhulst and Murdock [9], and the references quoted there. Recently many works extending and improving the averaging method for computing periodic solutions were presented, see for instance [1, 5, 4, 3]. Most of these results enhance the number of periodic solutions that can be detected by averaging method. Although few comments are made about the stability of these periodic solutions. To fill this gap the present work provides an strategy to determine the stability of the periodic orbits that bifurcate from of periodic orbits which form a manifold, or from points inside a continuum set that vanish some averaging functions, see Theorem 2. The detection of such bifurcations is possible by applying the Lyapunov–Schimdt reduction method over higher order averaging functions, see Theorem 1. This theorem was already used in [2] and [4] without the stability analysis.

We consider differential systems of the form

$$\dot{\mathbf{x}} = \mathbf{F}_0(t,\mathbf{x}) + \varepsilon\mathbf{F}_1(t,\mathbf{x}) + \varepsilon^2\mathbf{F}_2(t,\mathbf{x}) + \varepsilon^3\mathbf{F}_3(t,\mathbf{x}) + \varepsilon^4\mathbf{F}_4(t,\mathbf{x}) + \varepsilon^5\widetilde{\mathbf{F}}(t,\mathbf{x},\varepsilon), \quad (1)$$

with $\mathbf{x}$ in some open subset Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume $\mathbf{F}_i$ and $\widetilde{\mathbf{F}}$ for all $i = 1, 2, 3, 4$ are $T-$periodic in the variable t. Let $\mathbf{x}(t,\mathbf{z},0)$ be the solution of the unperturbed system

$$\dot{\mathbf{x}} = \mathbf{F}_0(t,\mathbf{x}),$$

such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. We define $M(t,\mathbf{z})$ the fundamental matrix of the linear differential system

$$\dot{\mathbf{y}} = \frac{\partial \mathbf{F}_0(t,\mathbf{x}(t,\mathbf{z},0))}{\partial \mathbf{x}}\mathbf{y},$$

such that $M(0,\mathbf{z})$ is the identity. The *displacement map* of system (1) is defined as

$$\mathbf{d}(\mathbf{z},\varepsilon) = \mathbf{x}(T,\mathbf{z},\varepsilon) - \mathbf{z}. \quad (2)$$

In order to have $\mathbf{d}(\mathbf{z},\varepsilon)$ well defined we assume that for $|\varepsilon| \neq 0$ sufficiently small

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