STABILITY OF PERIODIC ORBITS IN THE AVERAGING THEORY: APPLICATIONS TO LORENZ AND THOMAS' DIFFERENTIAL SYSTEMS

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Abstract. We study the kind of stability of the periodic orbits provided by higher order averaging theory. We apply these results for determining the $k-$hyperbolicity of some periodic orbits of the Lorenz and Thoma's differential system.

1. Introduction and statement of our main result

The averaging theory is a classical method for studying the solutions of the non-linear dynamical systems, and in particular their periodic solutions. For a general introduction to the averaging theory see the book of Sanders, Verhulst and Murdock [9], and the references quoted there. Recently many works extending and improving the averaging method for computing periodic solutions were presented, see for instance [1, 5, 4, 3]. Most of these results enhance the number of periodic solutions that can be detected by averaging method. Although few comments are made about the stability of these periodic solutions. To fill this gap the present work provides an strategy to determine the stability of the periodic orbits that bifurcate from of periodic orbits which form a manifold, or from points inside a continuum set that vanish some averaging functions, see Theorem 2. The detection of such bifurcations is possible by applying the Lyapunov–Schimdt reduction method over higher order averaging functions, see Theorem 1. This theorem was already used in [2] and [4] without the stability analysis.

We consider differential systems of the form

$$\dot{\mathbf{x}} = \mathbf{F}_0(t,\mathbf{x}) + \varepsilon \mathbf{F}_1(t,\mathbf{x}) + \varepsilon^2 \mathbf{F}_2(t,\mathbf{x}) + \varepsilon^3 \mathbf{F}_3(t,\mathbf{x}) + \varepsilon^4 \mathbf{F}_4(t,\mathbf{x}) + \varepsilon^5 \widetilde{\mathbf{F}}(t,\mathbf{x},\varepsilon), \quad (1)$$

with $\mathbf{x}$ in some open subset Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume $\mathbf{F}_i$ and $\widetilde{\mathbf{F}}$ for all $i = 1,2,3,4$ are $T-$periodic in the variable t. Let $\mathbf{x}(t,\mathbf{z},0)$ be the solution of the unperturbed system

$$\dot{\mathbf{x}} = \mathbf{F}_0(t,\mathbf{x}),$$

such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. We define $M(t,\mathbf{z})$ the fundamental matrix of the linear differential system

$$\dot{\mathbf{y}} = \frac{\partial \mathbf{F}_0(t,\mathbf{x}(t,\mathbf{z},0))}{\partial \mathbf{x}} \mathbf{y},$$

such that $M(0,\mathbf{z})$ is the identity. The *displacement map* of system (1) is defined as

$$\mathbf{d}(\mathbf{z},\varepsilon) = \mathbf{x}(T,\mathbf{z},\varepsilon) - \mathbf{z}. \quad (2)$$

In order to have $\mathbf{d}(\mathbf{z},\varepsilon)$ well defined we assume that for $|\varepsilon| \neq 0$ sufficiently small

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(H) there exists an open set $U \subset \Omega$ such that for all $\mathbf{z} \in U$ the unique solution $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ is defined on the interval $[0, t_{(\mathbf{z},\varepsilon)})$ with $t_{(\mathbf{z},\varepsilon)} > T$.

This hypothesis is always true when the unperturbed system has a manifold of T-periodic solutions. The standard method of averaging for finding periodic solutions consists in write the displacement map (2) as power series of ε in the following way

$$\mathbf{d}(\mathbf{z}, \varepsilon) = \mathbf{g}_0(\mathbf{z}) + \varepsilon \mathbf{g}_1(\mathbf{z}) + \varepsilon^2 \mathbf{g}_2(\mathbf{z}) + \varepsilon^3 \mathbf{g}_3(\mathbf{z}) + \varepsilon^4 \widetilde{\mathbf{g}}(\mathbf{z}, \varepsilon),$$

Where for $i = 0, 1, 2, 3, 4$ we have

$$\mathbf{g}_i(\mathbf{z}) = M(T, \mathbf{z})^{-1} \frac{\mathbf{y}_i(T, \mathbf{z})}{i!},$$

being

$$\begin{aligned}
\mathbf{y}_0(t, \mathbf{z}) =& \mathbf{x}(t, \mathbf{z}, 0) - \mathbf{z}, \\
\mathbf{y}_1(t, \mathbf{z}) =& M(t, \mathbf{z}) \int_0^t M(\tau, \mathbf{z})^{-1} \mathbf{F}_1(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathrm{d}\tau, \\
\mathbf{y}_2(t, \mathbf{z}) =& M(t, \mathbf{z}) \int_0^t M(\tau, \mathbf{z})^{-1} \bigg[2\mathbf{F}_2(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) + 2\frac{\partial \mathbf{F}_1}{\partial \mathbf{x}}(\tau, \mathbf{x}(\tau, \mathbf{x}, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \\
& + \frac{\partial^2 \mathbf{F}_0}{\partial \mathbf{x}^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^2 \bigg] \mathrm{d}\tau, \\
\mathbf{y}_3(t, \mathbf{z}) =& M(t, \mathbf{z}) \int_0^t M(\tau, \mathbf{z})^{-1} \bigg[6\mathbf{F}_3(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) + 6\frac{\partial \mathbf{F}_2}{\partial \mathbf{x}}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \\
& + 3\frac{\partial^2 \mathbf{F}_1}{\partial \mathbf{x}^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^2 + 3\frac{\partial \mathbf{F}_1}{\partial \mathbf{x}}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_2(\tau, \mathbf{z}) \\
& + 3\frac{\partial^2 \mathbf{F}_0}{\partial \mathbf{x}^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \odot \mathbf{y}_2(\tau, \mathbf{z}) + \frac{\partial^3 \mathbf{F}_0}{\partial \mathbf{x}^3}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^3 \bigg] \mathrm{d}\tau, \\
\mathbf{y}_4(t, \mathbf{z}) =& M(t, \mathbf{z}) \int_0^t M(\tau, \mathbf{z})^{-1} \bigg[24\mathbf{F}_4(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) + 24\frac{\partial \mathbf{F}_3}{\partial x}(\tau, \mathbf{x}(\tau, x, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \\
& + 12\frac{\partial^2 \mathbf{F}_2}{\partial x^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^2 + 12\frac{\partial \mathbf{F}_2}{\partial x}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_2(\tau, \mathbf{z}) \\
& + 12\frac{\partial^2 \mathbf{F}_1}{\partial x^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \odot \mathbf{y}_2(\tau, \mathbf{z}) + 4\frac{\partial^3 \mathbf{F}_1}{\partial x^3}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^3 \\
& + 4\frac{\partial \mathbf{F}_1}{\partial x}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_3(\tau, \mathbf{z}) + 3\frac{\partial^2 \mathbf{F}_0}{\partial x^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_2(\tau, \mathbf{z})^2 \\
& + 4\frac{\partial^2 \mathbf{F}_0}{\partial x^2}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z}) \odot \mathbf{y}_3(\tau, \mathbf{z}) \\
& + 6\frac{\partial^3 \mathbf{F}_0}{\partial x^3}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^2 \odot \mathbf{y}_2(\tau, \mathbf{z}) + \frac{\partial^4 \mathbf{F}_0}{\partial x^4}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0)) \mathbf{y}_1(\tau, \mathbf{z})^4 \bigg] \mathrm{d}\tau.
\end{aligned}$$

The functions $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ and $\mathbf{g}_4$ will be called here the *averaged functions* of order 1, 2, 3 and 4 respectively of system (1).

We say that system (1) has a periodic solution bifurcating from the point $\mathbf{z}_0$ if there exists a branch of solutions $\mathbf{z}(\varepsilon)$ for the displacement function such that $\mathbf{d}(\mathbf{z}(\varepsilon), \varepsilon) = \mathbf{0}$ and $\mathbf{z}(0) = \mathbf{z}_0$.

Now we shall present our result about the existence and stability of the periodic solutions of system (1). The methodology used here was introduced for studying differential systems such that the unperturbed part has a sub-manifold of T-periodic solutions, see for instance [1] and [4]. The main difference of this work with the previous ones is that the first nonzero averaged function vanishes over a graph.

Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\pi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the projections onto the first m coordinates and onto the last $n-m$ coordinates, respectively. For a point $\mathbf{z} \in U$ we also consider $\mathbf{z} = (a, b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Consider the graph

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\} \subset U \tag{3}$$

such that $m < n$, V is an open set of $\mathbb{R}^m$ and $\beta : \overline{V} \to \mathbb{R}^{n-m}$ is a $\mathcal{C}^4$ function.

The next theorem provides sufficient conditions for the existence of periodic solutions in the differential system (1). This theorem was proved in [3] here we also provide an scheme of its proof. We need this theorem for the statement of our main result in Theorem 2.

Theorem 1. *Let $r \in \{0, 1, 2\}$ such that r is the first subindex such that $\mathbf{g}_r \not\equiv 0$. In addition to hypothesis (H) assume that*

- (i) *the averaged function $\mathbf{g}_r$ vanishes on the graph (3). That is $\mathbf{g}_r(\mathbf{z}_\alpha) = \mathbf{0}$ for all $\alpha \in \overline{V}$, and*
- (ii) *the Jacobian matrix*

$$D\mathbf{g}_r(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix}$$

where $\Lambda_\alpha = D_a \pi g_r(z_\alpha)$, $\Gamma_\alpha = D_b \pi g_r(z_\alpha)$, $B_\alpha = D_a \pi^\perp g_r(z_\alpha)$ and $\Delta_\alpha = D_b \pi^\perp g_r(z_\alpha)$, satisfies that $\det(\Delta_\alpha) \neq 0$ for all $\alpha \in \overline{V}$.

We define the functions

$$\begin{aligned} f_1(\alpha) =& -\Gamma_\alpha \Delta_\alpha^{-1} \pi^\perp \mathbf{g}_{r+1}(\mathbf{z}_\alpha) + \pi \mathbf{g}_{r+1}(\mathbf{z}_\alpha), \\ f_2(\alpha) =& \frac{1}{2}\Gamma_\alpha \gamma_2(\alpha) + \frac{1}{2}\frac{\partial^2 \pi \mathbf{g}_r}{\partial b^2}(\mathbf{z}_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi \mathbf{g}_{r+1}}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi \mathbf{g}_{r+2}(\mathbf{z}_\alpha), \\ \gamma_1(\alpha) =& -\Delta_\alpha^{-1} \pi^\perp \mathbf{g}_{r+1}(\mathbf{z}_\alpha), \\ \gamma_2(\alpha) =& -\Delta_\alpha^{-1}\left(\frac{\partial^2 \pi^\perp \mathbf{g}_r}{\partial b^2}(\mathbf{z}_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp \mathbf{g}_{r+1}}{\partial b}(\mathbf{z}_\alpha)\gamma_1(\alpha) + 2\pi^\perp \mathbf{g}_{r+2}(\alpha)\right). \end{aligned} \tag{4}$$

Then the following statements hold.

- (a) *If there exists $\alpha^* \in V$ such that $f_1(\alpha^*) = 0$ and $\det(Df_1(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small there is an initial condition $\mathbf{z}(\varepsilon) \in U$ such that $\mathbf{z}(0) = \mathbf{z}_{\alpha^*}$ and the solution $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$ of system (1) is T-periodic.*
- (b) *Assume that $f_1 \equiv 0$. If there exists $\alpha^* \in V$ such that $f_2(\alpha^*) = 0$ and $\det(Df_2(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small there is an initial condition $\mathbf{z}(\varepsilon) \in U$ such that $\mathbf{z}(0) = \mathbf{z}_{\alpha^*}$ and the solution $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$ of system (1) is T-periodic.*

Theorem 1 shows that the function f_1 and f_2 provides sufficient conditions for the existence of periodic solutions of the differential system (1).

For periodic solutions detected by statement (a) of Theorem 1 the next result reveals how the higher order function f_2 can be used for determining the stability of the periodic solution $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$.

Theorem 2. *Consider r, Γ_α, Δ_α, f_1 and f_2 as defined in Theorem 1 and the Jacobian matrices $D\mathbf{g}_r(\mathbf{z}) = (p_{ij}(\mathbf{z}))$ and $D\mathbf{g}_{r+1}(\mathbf{z}) = (q_{ij}(\mathbf{z}))$. Assume that there* *exists* $\alpha^* \in V$ *such that* $f_1(\alpha^*) = 0$ *and* $\det(Df_1(\alpha^*)) \neq 0$. *We define the matrix function*

$$A(\varepsilon) = A_0 + \varepsilon A_1, \tag{5}$$

where

$$A_0 = Dg_r(\mathbf{z}_{\alpha^*}), \tag{6}$$
$$A_1 = \left(Dp_{ij}(\mathbf{z}_{\alpha^*}).\mathbf{z}_1 + q_{ij}(\mathbf{z}_{\alpha^*})\right), \tag{7}$$
$$\mathbf{z}_1 = \left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*),\, D\beta(\alpha^*)\left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*)\right) + \gamma_1(\alpha^*)\right). \tag{8}$$

Then the periodic orbit $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$ *has the same type of hyperbolic stability as the matrix* $A(\varepsilon)$ *provided that:*

(s_1) A_0 *has no multiple eigenvalues on the imaginary axis, and*
(s_2) *there exists* $c > 0$ *such that every eigenvalue* $\lambda(\varepsilon)$ *of* $A(\varepsilon)$ *satisfies* $|\mathrm{Re}(\lambda(\varepsilon))| > c\varepsilon$ *for all sufficiently small* $|\varepsilon| > 0$.

The same class of result can be obtained for periodic orbits detected by statement (b) of Theorem 1 using the bifurcation function of order 3. The expressions of such functions are explicitly given in [3].

2. Applications

Lorenz differential system. Consider the differential system

$$\begin{aligned} \dot{x} &= a(x - y), \\ \dot{y} &= x(b - z) - y, \\ \dot{z} &= x\,y - cz, \end{aligned} \tag{9}$$

with a, b, c being real coefficients. In recent publication [2] the authors have found a periodic orbit bifurcating from the origin of system (9), see Figure 2. The next theorem completes this work giving the stability characterization of that periodic solution.

Theorem 3. *Let* $a = -1 + a_2\varepsilon^2$ *and* $c = c_1\varepsilon$. *Assume that* $b > 1$, $a_2 < 0$, $c_1 \neq 0$ *and* $|\varepsilon| \neq 0$ *sufficiently small. Then the Lorenz differential system* (9) *has a periodic orbit bifurcating from the origin. Furthermore for* $c_1 > 0$ *this periodic orbit is an attractor, otherwise for* $c_1 < 0$ *the periodic orbit has a stable manifold formed by two topological cylinders and an unstable manifold formed by two topological cylinders.*

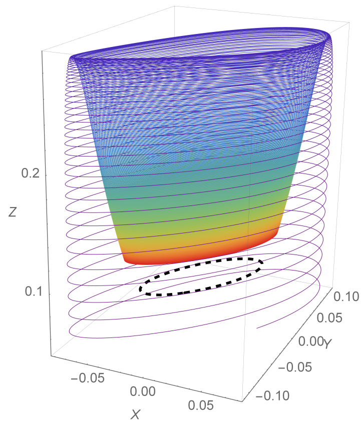

Figure 1. Solution of system (9) starting at $(0.05, -0.01, 0.05)$ being attracted by the stable periodic orbit (dashed courve) founded by Theorem 1. The parameters of the system are $a_2 = -2$, $b = 2$ $c_1 = 1$ and $\varepsilon = 1/100$.

Theorem 3 is proved in section 3 using Theorems 1 and 2.

Thomas' systems. A circulant system is a differential system defined by a function $f(x, y, z)$ having the variables cyclically symmetric according to

$$\begin{aligned} \dot{x} &= f(x, y, z), \\ \dot{y} &= f(y, z, x), \\ \dot{x} &= f(z, x, y), \end{aligned}$$

where the function $f(u, v, w)$ is fixed and the variables are rotated. In 1999 René Thomas propose two circulant systems having cyclic symmetry

$$\begin{aligned} \dot{x} &= \sin y - \beta x, \\ \dot{y} &= \sin z - \beta y, \\ \dot{z} &= \sin x - \beta z, \end{aligned} \qquad (10)$$

$$\begin{aligned} \dot{x} &= -bx + ay - y^3, \\ \dot{y} &= -by + az - z^3, \\ \dot{z} &= -bz + ax - x^3. \end{aligned} \qquad (11)$$

System (10) is defeined by the function $f(u, v, w) = -au + \sin v$ and system (11) is defined by $f(u, v, w) = -au + bv - v^3$. The chaotic behaviour generated by these systems was presented in [12], system (10) was also studied by Sprott and Chlouverakis in [11]. System (10) is sometimes called Thomas' system, see for instance [10, Chapter 3]. The next results give sufficient conditions for the existence of periodic solutions on these differential systems.

One can check that the origin is an equilibrium point of system (10), and that it has the eigenvalues $1 - \beta$, $(-1 - 2\beta - i\sqrt{3})/2$ and $(-1 - 2\beta + i\sqrt{3})/2$. When $\beta = -1/2$ the origin has a pair of complex eigenvalues on the imaginary axis and the bifurcation of a periodic orbit occurs.

Theorem 4. *Let* $\beta = -1/2 + \beta_1\varepsilon + \beta_2\varepsilon^2$ *where* $\beta_i \in \mathbb{R}$ *for* $i = 1, 2$. *For* $\varepsilon > 0$ *sufficiently small and* $\beta_1 > 0$ *the differential system* (10) *has an isolated periodic solution bifurcating from the origin.*

Theorem 4 is proved in section 3 using Theorems 1 and 2 taking $r = 0$. System (11) has 27 stead states but we will be interested into the pair symmetric equilibrium points $\mathbf{P}_\pm = \pm\left(\sqrt{a-b}, \sqrt{a-b}, \sqrt{a-b}\right)$. Taking $a = 5\sqrt{3}\omega/6$ and $b = \sqrt{3}\omega/3$ with $\omega > 0$, these equilibrium points have the eigenvalues $-\sqrt{3}\omega$ and $\pm\omega i$. The next theorems show that periodic orbits are born at $\mathbf{P}_-$ and $\mathbf{P}_+$.

Theorem 5. *Let $a = 5\sqrt{3}\omega/6+\varepsilon a_1$, $b = \sqrt{3}\omega/3+\varepsilon b_1$ with $\omega > 0$ and $(5b_1-2a_1) < 0$. Then for $\varepsilon > 0$ sufficiently small the differential system* (11) *has two periodic solutions*

$$\begin{aligned}\phi_\pm(t,\varepsilon) =&\mathbf{P}_\pm + \sqrt{\varepsilon}\Big(2e^{2\sqrt{3}\pi}\xi\cos(t\omega), \frac{e^{2\sqrt{3}\pi}}{3}\xi\left(3\sin(t\omega)-\sqrt{3}\cos(t\omega)\right),\\ &-\frac{1}{3}e^{2\sqrt{3}\pi}\xi\left(3\sin(t\omega)+\sqrt{3}\cos(t\omega)\right)\Big) + \mathcal{O}(\varepsilon),\end{aligned} \tag{12}$$

such that $\phi_+(t,\varepsilon)$ bifurcates from $\mathbf{P}_+$ and $\phi_-(t,\varepsilon)$ bifurcates from $\mathbf{P}_-$. Here $\xi = \sqrt{\dfrac{\pi(5b_1-2a_1)}{-3e^{4\sqrt{3}\pi}\left(\sqrt{3}-5\pi\right)+6\sqrt{3}e^{2\sqrt{3}\pi}-3\sqrt{3}}}$.

The periodic orbit analytically found in Theorem 5 was detected numerically by Thomas in [12], he also shows for specific values of a and b that these periodic solutions give born to a strange attractors after a cascade of doubling. The following figures illustrate this phenomena. Here $a_1 = 6$, $b_1 = 1$ and $\omega = 1$ the time interval is from 0 to 1000. Figure 2 shows the solution starting at $(-0.8, -0.8, -0.45)$ being attracted by the periodic orbit $\phi_-(t,\varepsilon)$, see equation (12). As we increase ε the periodic orbit grows in size and complexity, see Figures 3, 4. The approximation to the periodic orbit provided by (12) can be seen as a dashed curve. Figures 5, 6 and 7 shows the appearance of the strange attractor as ε increase.

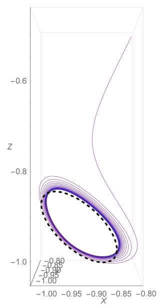

Figure 2. $\varepsilon = 1/250$

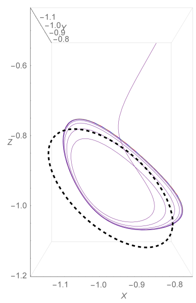

Figure 3. $\varepsilon = 1/50$

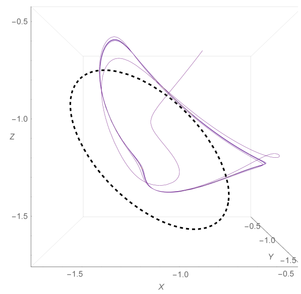

Figure 4. $\varepsilon = 1/8$

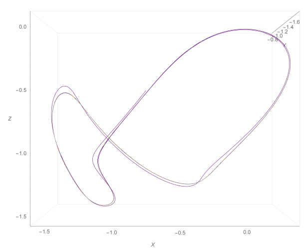

Figure 5. $\varepsilon = 1/6$

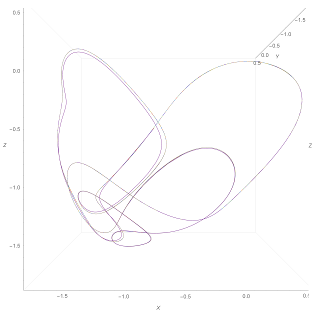

Figure 6. $\varepsilon = 1/5$

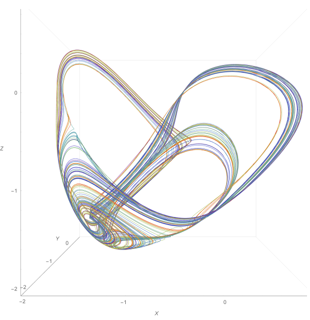

Figure 7. $\varepsilon = 1/4$

3. Proofs

Proof of Theorem 1. For a detailed proof see [3], but for the sake of completeness we present here the ideas of the proof. Define the function $\mathbf{g}(\mathbf{z},\varepsilon) = \mathbf{d}(\mathbf{z},\varepsilon)/\varepsilon^r$

where $\mathbf{d}(\mathbf{z}(\varepsilon),\varepsilon)$ is the displacement function and r is defined as in the statement of the theorem. We have that

$$\mathbf{g}(\mathbf{z}(\varepsilon),\varepsilon)) = \mathbf{g}_r(\mathbf{z}) + \sum_{i=1}^{3-r} \mathbf{g}_{r+i}(\mathbf{z})\varepsilon^i + \mathcal{O}\left(\varepsilon^{4-r+1}\right).$$

From here the proof is just apply Lemma 3 of [4]. Here we present an sketch of the proof of this lemma, more details can be obtained in the Section 2 of [4]. The first step is to write $\mathbf{g} = (\pi\mathbf{g}, \pi^\perp\mathbf{g})$. Using $\pi^\perp\mathbf{g}$ we define de function

$$\begin{aligned}\delta^\perp : \mathbb{R}^m \times [-\varepsilon_0, \varepsilon_0] \times \mathbb{R}^{n-m} &\to \mathbb{R}^{n-m}\\ ((a,\varepsilon),b) &\mapsto \pi^\perp\mathbf{g}((a,b),\varepsilon) = \pi^\perp\mathbf{g}_r(a,b) + \varepsilon\pi^\perp\mathbf{g}_{r+1}(a,b) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Then from hypothesis (i) and (ii) we have that $\delta^\perp((\alpha,0),\beta(\alpha)) = \pi^\perp\mathbf{g}_r(\alpha,\beta(\alpha)) = 0$ and $D_b\big(\pi^\perp\delta\big)((\alpha,0),\beta(\alpha)) = \Delta_\alpha$. Since $\det(\Delta_\alpha) \neq 0$, we apply the Implicit Function Theorem obtaining a $\mathcal{C}^4$ function $\overline{\beta} : U\times(-\varepsilon_1,\varepsilon_1) \to \mathbb{R}^{n-m}$ where $U\times(-\varepsilon_1,\varepsilon_1)$ is a neighbourhood of $\overline{V}\times\{0\}$ such that $\overline{\beta}(\alpha,0) = \beta(\alpha)$ and $\pi^\perp\mathbf{g}((\alpha,\overline{\beta}(\alpha,\varepsilon)),\varepsilon) = \delta^\perp(\alpha,\overline{\beta}(\alpha,\varepsilon),\varepsilon) = 0$. Mainly,

$$\overline{\beta}(\alpha,\varepsilon) = \beta(\alpha) + \sum_{i=1}^{3-r} \gamma_i(\alpha)\varepsilon^i + \mathcal{O}(\varepsilon^{4-r}), \tag{13}$$

and the functions γ_i for $i = 1,2,3$, are shown in equation (15) of [4]. The function $\beta(\alpha)$ is given in (3). Now for all $\alpha \in \overline{V}$ we consider the function $\delta(\alpha,\varepsilon) = \pi\mathbf{g}(\alpha,\overline{\beta}(\alpha,\varepsilon),\varepsilon)$. Writing this function as a power series of ε we obtain

$$\delta(\alpha,\varepsilon) = \sum_{i=1}^{3-r} \varepsilon^i f_i(\alpha) + \mathcal{O}(\varepsilon^{4-r}). \tag{14}$$

We use the auxiliary function

$$\mathcal{F}(\alpha,\varepsilon) = \frac{\delta(\alpha,\varepsilon)}{\varepsilon} = f_1(\alpha) + \sum_{i=2}^{3-r} \varepsilon^{i-1} f_i(\alpha) + \mathcal{O}(\varepsilon^{3-r}), \tag{15}$$

for studying the branches of zeros of (14).If there exits $\alpha^* \in \overline{V}$ such that $f_1(\alpha^*) = 0$ and $\det\left(Df_1(\alpha^*)\right) \neq 0$ then by the Implicit Function Theorem we have that there exists a branch of zeros such that $\mathcal{F}(\alpha(\varepsilon),\varepsilon) = 0$ and

$$\alpha(\varepsilon) = \alpha^* + \mathcal{O}(\varepsilon). \tag{16}$$

Consequently $\pi\mathbf{g}(\alpha(\varepsilon),\overline{\beta}(\alpha(\varepsilon),\varepsilon),\varepsilon) = 0$ and $\pi^\perp\mathbf{g}(\alpha(\varepsilon),\overline{\beta}(\alpha(\varepsilon),\varepsilon),\varepsilon) = 0$, then $\mathbf{g}(\alpha(\varepsilon),\overline{\beta}(\alpha(\varepsilon),\varepsilon),\varepsilon) = 0$. This means that $\mathbf{z}(\varepsilon) = \big(\alpha(\varepsilon),\overline{\beta}(\alpha(\varepsilon),\varepsilon)\big)$ is a branch of zeros to the displacement function, i. e., $\mathbf{d}(\mathbf{z}(\varepsilon),\varepsilon) = \varepsilon^r\mathbf{g}(\mathbf{z}(\varepsilon),\varepsilon) = 0$. Thus for $|\varepsilon| > 0$ sufficiently small $\mathbf{x}(t,\mathbf{z}(\varepsilon),\varepsilon)$ is a T-periodic solution of system (1). This concludes the proof of statement (a) of Theorem 1. The proof of statement (b) is analogous. □

A fundamental notion in qualitative theory of differential equations is hyperbolicity. A constant matrix will be called *hyperbolic* if its eigenvalues lie out of the imaginary axis, in which case its *index* is the number of eigenvalues in the right half-plane. Consider a matrix function $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + \varepsilon^{(k-1)}A_k$ depending on a parameter ε. If A_0 is hyperbolic of index i, then one can see that for $\varepsilon > 0$ sufficiently small $A(\varepsilon)$ will be hyperbolic with the same index i.

If A_0 is not hyperbolic the placement of the eigenvalues of $A(\varepsilon)$ may be hard to determine. To deal with this problem we use a method introduced by Murdock and Robinson, see [7] and [8]. The matrix $A(\varepsilon)$ is called *k-hyperbolic of index i* if for every smooth matrix function $B(\varepsilon)$ there exists an $\varepsilon_0 > 0$ such that $A(\varepsilon) + \varepsilon^k B(\varepsilon)$

is hyperbolic of index i for all ε in the interval $0 < \varepsilon < \varepsilon_0$. The next result will be needed for proving Theorem 2.

Assume that there exists a matrix function $S(\varepsilon)$ that block diagonalizes $A(\varepsilon)$ into its left, right and center blocks $L(\varepsilon)$, $C(\varepsilon)$, $R(\varepsilon)$ which for $\varepsilon = 0$ have their eigenvalues respectively in the left half-plane, on the imaginary axis, and in the right half-plane. Thus

$$S(\varepsilon)^{-1}A(\varepsilon)S(\varepsilon) = \begin{bmatrix} L(\varepsilon) & 0 & 0 \\ 0 & C(\varepsilon) & 0 \\ 0 & 0 & R(\varepsilon) \end{bmatrix}.$$

Theorem 6 ([6, Theorem 5.7])**.** *Let $C(\varepsilon)$ be the center block of $A(\varepsilon)$ and let its size be $m \times m$. Then $A(\varepsilon)$ is k-hyperbolic provided that:*

(a) *A_0 has no multiple eigenvalues on the imaginary axis.*
(b) *There exists $c > 0$ such that every eigenvalue $\lambda(\varepsilon)$ of $C(\varepsilon)$ satisfies $|\mathbf{Re}\,\lambda(\varepsilon)| \geq c\, e^{k-1}$ for all small ε.*

Proof of Theorem 2. The statement that $r \in \{0, 1\}$ allows to obtain more information about functions (13) and (15). We can write

$$\begin{aligned} \overline{\beta}(\alpha, \varepsilon) =& \beta(\alpha) + \varepsilon\gamma_1(\alpha) + \varepsilon^2\gamma_2(\alpha) + \mathcal{O}(\varepsilon^{4-r}), && (17)\\ \mathcal{F}(\alpha, \varepsilon) =& f_1(\alpha) + \varepsilon f_2(\alpha) + \mathcal{O}(\varepsilon^{3-r}), && (18) \end{aligned}$$

where f_1 is defined in statement (iii) of Theorem 1 and f_2, γ_1 and γ_2 are defined in (4). Using the Implicit Function Theorem in the (18) we have that the branch of zeros (16) writes

$$\alpha(\varepsilon) = \alpha(0) + \varepsilon\alpha'(0) + \mathcal{O}(\varepsilon^2) = \alpha^* + \varepsilon\left(-Df_1^{-1}(\alpha^*)f_2(\alpha^*)\right) + \mathcal{O}(\varepsilon^2), \quad (19)$$

where $Df_1^{-1}(\alpha^*)$ is the inverse of the Jacobian matrix of the function $f_1(\alpha)$ at the point α^*. Substituting (19) into (17) and expanding the result in Taylor's series around $\varepsilon = 0$ we obtain

$$\begin{aligned} \overline{\beta}(\alpha(\varepsilon), \varepsilon) &= \beta(\alpha(0)) + \varepsilon\left(D\beta(\alpha(0))\alpha'(0) + \gamma_1(\alpha(0))\right) + \mathcal{O}(\varepsilon^2) \\ &= \beta(\alpha^*) + \varepsilon\left(D\beta(\alpha^*)\left(-Df_1^{-1}(\alpha^*)f_2(\alpha^*)\right) + \gamma_1(\alpha(0))\right) + \mathcal{O}(\varepsilon^2). \quad (20) \end{aligned}$$

Here $D\beta(\alpha^*)$ is the Jacobian matrix of function $\beta(\alpha)$ at α^*. From (19) and (20) we have that

$$\mathbf{z}(\varepsilon) = \left(\alpha(\varepsilon), \overline{\beta}(\alpha(\varepsilon), \varepsilon)\right) = \mathbf{z}_{\alpha^*} + \varepsilon\mathbf{z}_1 + \mathcal{O}(\varepsilon^2), \quad (21)$$

with $\mathbf{z}_{\alpha^*} = (\alpha^*, \beta(\alpha^*))$ and $\mathbf{z}_1$ is defined in (8).

Using (21) we can write the Jacobian matrix of the displacement function at $\mathbf{z}(\varepsilon)$ as a power series of ε around $\varepsilon = 0$ as

$$\frac{\partial \mathbf{d}(\mathbf{z}(\varepsilon), \varepsilon)}{\partial \mathbf{z}} = A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2), \quad (22)$$

a classical result about ordinary differential equations says that when (22) is a hyperbolic matrix, the periodic solution $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$ will be hyperbolic with the same kind of stability. This is also referred as linear stability. Thus the proof of the theorem follows from applying Theorem 6 in the 1-jet (5) observing that hypothesis (s_1) and (s_2) are equivalent with the hypothesis (a') and (b') respectively. Thus the matrix is $2-$hyperbolic and the theorem is proved. $\square$

Proof of Theorem 3. The existence of such periodic orbit is proved in Theorem 4 of [2]. Following the ideas of this proof we see that, after some changes of variables, system (9) can be put into the normal form for applying Theorem 1,

$$\dot{\mathbf{z}} = \varepsilon \mathbf{F}_1\big(\mathbf{z},\theta\big) + \varepsilon^2 \mathbf{F}_2\big(\mathbf{z},\theta\big) + \varepsilon^3 \mathbf{F}_2\big(\mathbf{z},\theta\big) + \mathcal{O}(\varepsilon^4)$$

given by equation (22) of [2], with $\mathbf{z} = (\rho, z)$ and the derivative with respect to θ. Thus calculating the higher order averaging functions of this system for $i = 0,1,2,3$ we have $\mathbf{g}_i(\mathbf{z}) = (g_{i1}(\mathbf{z}), g_{i2}(\mathbf{z}))$ where $\mathbf{g}_0(\mathbf{z}) \equiv 0$ and

$$\begin{aligned}
g_{11}(\mathbf{z}) =& 0,\\
g_{12}(\mathbf{z}) =& \frac{\pi\left(\rho^2 - 2c_1 z\right)}{\omega},\\
g_{21}(\mathbf{z}) =& -\frac{\pi\rho\left(8a_2\omega^2 - 4c_1 z + 3\rho^2\right)}{8\omega^3},\\
g_{22}(\mathbf{z}) =& \frac{\pi\left(\rho^2(c_1\omega(\omega-2\pi)+3z) + 2c_1 z(2\pi c_1\omega - z)\right)}{2\omega^3},\\
g_{31}(\mathbf{z}) =& -\frac{\pi\rho\left(4z\left(2a_2\omega^2 + 2\pi c_1^2\omega - 3c_1 z\right) + \rho^2(c_1\omega(3\omega-4\pi)+15z)\right)}{16\omega^5},\\
g_{32}(\mathbf{z}) =& \frac{\pi}{96\omega^5}\Big(9\rho^4\omega(4\pi-5\omega) - 8c_1 z\left(12a_2\omega^4 + 16\pi^2c_1^2\omega^2 - 36\pi c_1\omega z + 9z^2\right) + 4\rho^2\\
&\left(3c_1\omega(9\omega-28\pi)z + 45z^2\right) - 2\omega^2\left(6a_2\omega(\omega+2\pi) + c_1^2\left(6\pi\omega - 8\pi^2 + 3\right)\right)\Big).
\end{aligned}$$

Thus we can calculate the functions $f_i(\alpha)$ for $i = 1,2$ with respect to the averaging functions above and the graph

$$\mathcal{Z} = \left\{\mathbf{z}_\alpha = \left(\alpha, \beta(\alpha) = \frac{\alpha^2}{2c_1}\right) : \alpha > 0\right\},$$

obtaining

$$f_1(\alpha) = -\frac{\pi\alpha\left(8a_2\omega^2 + \alpha^2\right)}{8\omega^3} \quad\text{and}\quad f_2(\alpha) = -\frac{\pi\alpha^3\left(2\omega^2\left(4a_2 + c_1^2\right) + 5\alpha^2\right)}{32c_1\omega^5}.$$

By the hypothesis of Theorem 3 one can check that $\alpha^* = 2\omega\sqrt{-2a_2}$ is a simple zero of function $f_1(\alpha)$. Then we can apply Theorem 2 with $r = 1$. By (21) we can write the initial point of the periodic solution as $\mathbf{z}(\varepsilon) = \mathbf{z}_{\alpha^*} + \varepsilon\mathbf{z}_1$ with

$$\mathbf{z}_1 = \left(\frac{(16a_2 - c_1^2)\omega\sqrt{-2a_2}}{2c_1}, 4a_2\omega^2\left(\frac{12a_2}{c_1^2} - 1\right)\right),$$

and the matrix (5) becomes

$$A(\varepsilon) = \begin{pmatrix} 0 & 0\\ 4\pi\sqrt{-2a_2} & -\dfrac{2c_1\pi}{\omega}\end{pmatrix} + \varepsilon\begin{pmatrix} \dfrac{6a_2\pi}{\omega} & \dfrac{\sqrt{-2a_2}c_1\pi}{\omega^2}\\ \dfrac{\pi\sqrt{-2a_2}}{\omega c_1}\left(c_1^2(\omega-4\pi) - 8a_2\omega\right) & \dfrac{2\pi(c_1^2\pi - 2a_2\omega)}{\omega^2}\end{pmatrix}.$$

The matrix $A(\varepsilon)$ has the two distinct eigenvalues

$$\lambda_1 = -\frac{2c_1\pi}{\omega} + \varepsilon\left(\frac{2c_1\pi}{\omega}\right)^2 + \mathcal{O}(\varepsilon^2) \text{ and } \lambda_2 = \varepsilon\frac{2a_2\pi}{\omega} + \mathcal{O}(\varepsilon^2).$$

As a_2 is negative by hypothesis, we have that for $\varepsilon > 0$ sufficiently small if $c_1 > 0$, $\mathbf{Re}(\lambda_1) < \mathbf{Re}(\lambda_2) < 0$ consequently the periodic orbit is an attractor. Otherwise, if $c_1 < 0$, $\mathbf{Re}(\lambda_2) < 0 < \mathbf{Re}(\lambda_1)$ then the periodic orbit has a stable manifold formed by two topological cylinders, and an unstable manifold formed by two topological cylinders. □

Proof of Theorem 4. Using the change of variables $(X,Y,Z)=\sqrt{\varepsilon}\big(x+z,(-x-\sqrt{3}y+2z)/2,(-x+\sqrt{3}y+2z)/2\big)$ the differential system (10) becomes

$$
\begin{aligned}
\dot X =&X\left(\frac{1}{2}-\beta_2\varepsilon^2\right)-\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{3\sqrt{\varepsilon}}+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{3\sqrt{\varepsilon}}\\
&+\frac{2\sin\left(\sqrt{\varepsilon}\left(X+\sqrt{3}Y-2Z\right)/2\right)}{3\sqrt{\varepsilon}},\\
\dot Y =&Y\left(\frac{1}{2}-\beta_2\varepsilon^2\right)+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{\varepsilon\sqrt{3}}-\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{\sqrt{\varepsilon}\sqrt{3}},\\
\dot Z =&Z\left(\frac{1}{2}-\beta_2\varepsilon^2\right)-\frac{\sin\left(\sqrt{\varepsilon}\left(X+\sqrt{3}Y-2Z\right)/2\right)}{3\sqrt{\varepsilon}}+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{3\sqrt{\varepsilon}}\\
&+\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{3\sqrt{\varepsilon}},
\end{aligned}
\tag{23}
$$

we remark that for all $\delta\in\mathbb{R}$ the function $\sin(\delta\, w)/\delta$ is well defined and

$$\lim_{\delta\to 0}\frac{\sin(\delta\, w)}{\delta}=w.$$

Thus the equation above can also be written as

$$
\begin{aligned}
\dot X =&-\frac{\sqrt{3}}{2}Y+\frac{\varepsilon}{16}\Big(X^3+X^2\left(\sqrt{3}Y+2Z\right)+X\left(Y^2-4\sqrt{3}YZ+4\left(Z^2-4\beta\right)\right)\\
&+Y\left(\sqrt{3}Y^2-2YZ+4\sqrt{3}Z^2\right)\Big)+\mathcal{O}(\varepsilon^2),\\
\dot Y =&\frac{\sqrt{3}}{2}X+\frac{\varepsilon}{16}\Big(-\sqrt{3}X^3+X^2\left(Y-2\sqrt{3}Z\right)-X\left(\sqrt{3}Y^2+4YZ+4\sqrt{3}Z^2\right)\\
&+Y\left(Y^2+2\sqrt{3}YZ+4\left(Z^2-4\beta\right)\right)\Big)+\mathcal{O}(\varepsilon^2),\\
\dot Z =&\frac{3}{2}Z+\frac{\varepsilon}{24}\Big(-X^3-6X^2Z+3XY^2-2Z\left(3Y^2+2\left(6\beta+Z^2\right)\right)\Big)+\mathcal{O}(\varepsilon^2).
\end{aligned}
$$

In order to put the differential system (23) into the normal form for applying the averaging theory we consider the cylindrical change of variables $(X,Y,Z)=\big(\rho\cos\theta,\rho\sin\theta,w\big)$ with $\rho>0$. Then we check that $\dot\theta=\sqrt{3}/2+\mathcal{O}(\varepsilon^2)$ for $|\varepsilon|\neq 0$ sufficiently small. Thus taking θ as the new independent variable we obtain the differential system

$$\dot{\mathbf{z}}=\mathbf{F}_0\big(\mathbf{z},\theta\big)+\varepsilon\mathbf{F}_1\big(\mathbf{z},\theta\big)+\varepsilon^2\mathbf{F}_2\big(\mathbf{z},\theta\big)+\mathcal{O}(\varepsilon^3), \tag{24}$$

with $\mathbf{z}=(\rho,w)$, $\mathbf{F}_0\big(\mathbf{z},\theta\big)=\big(0,\sqrt{3}w\big)$, and $\mathbf{F}_i\big(\mathbf{z},\theta\big)=\big(F_{i1}\big(\mathbf{z},\theta\big),F_{i2}\big(\mathbf{z},\theta\big)\big)$ for $i=1,2$, where

$$
\begin{aligned}
F_{11}\big(\mathbf{z},\theta\big)=&\frac{\rho}{8\sqrt{3}}\left(\rho^2+2\rho w\left(\cos(3\theta)-\sqrt{3}\sin(3\theta)\right)+4\left(w^2-4\beta\right)\right),\\
F_{12}\big(\mathbf{z},\theta\big)=&\frac{1}{72}\Big(w\left(\sqrt{3}\left(-48\beta-3\rho^2+28w^2\right)+18rw\sin(3\theta)\right)\\
&-2\sqrt{3}\rho\cos(3\theta)\left(\rho^2-9w^2\right)\Big),\\
F_{21}\big(\mathbf{z},\theta\big)=&\frac{\rho}{5760}\Big(\rho\Big(-30w\sin(3\theta)\left(32\beta+\rho^2+8w^2\right)+10\sqrt{3}w\cos(3\theta)\\
&\left(-96\beta+7\rho^2+40w^2\right)+3\rho\sin(6\theta)\left(\rho^2-40w^2\right)\\
&-\sqrt{3}\rho\cos(6\theta)\left(\rho^2-120w^2\right)\Big)+20\sqrt{3}\left(-192\beta_2+\rho^4+6\rho^2\left(w^2-4\beta\right)\right.\\
&\left.+20w^4-96\beta w^2\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{22}(\mathbf{z},\theta) =& \frac{1}{5760\sqrt{3}}\Big(-12w\left(960\beta_2+5\rho^4+120\beta\left(\rho^2+4w^2\right)-50\rho^2w^2-228w^4\right) \\
&-30\rho\cos(3\theta)\left(\rho^4+3\rho^2w^2-104w^4+96\beta w^2\right) \\
&+\rho w\Big(\cos(6\theta)\left(360\rho w^2-69\rho^3\right)+2\sqrt{3}\sin(3\theta)\left(\cos(3\theta)\left(360\rho w^2-23\rho^3\right)\right. \\
&+\left.5w\left(-96\beta+3\rho^2+104w^2\right)\right)\Big)\Big).
\end{aligned}
$$

System (24) is 2π-periodic and it is into the normal form for applying Theorem 1. Furthermore for an initial condition $\mathbf{z}_0=(\rho_0,w_0)$ the solution of the unperturbed differential system corresponding to (24) is given by $\Phi(\theta,\mathbf{z})=\left(\rho_0, w_0e^{\sqrt{3}\theta}\right)$. Then we consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. Clearly for $\mathbf{z}_\alpha\in\mathcal{Z}$ the solution $\Phi(\theta,\mathbf{z}_\alpha)$ can be assumed 2π-periodic, and therefore the differential system (24) satisfies the hypothesis (H). Moreover the fundamental matrix of the variational differential system along $\Phi(\theta,\mathbf{z}_\alpha)$ is

$$
M(\theta,\mathbf{z}_\alpha)=\begin{pmatrix}1 & 0\\ 0 & e^{\sqrt{3}\theta}\theta\end{pmatrix}.
$$

Computing the averaging functions we obtain $\mathbf{g}_0(\mathbf{z})=\left(0,(e^{2\pi\sqrt{3}}-1)w\right)$ and $\mathbf{g}_i(\mathbf{z})=(g_{i1}(\mathbf{z}),g_{i2}(\mathbf{z}))$ for $i=1,2$ where

$$
\begin{aligned}
g_{11}(\mathbf{z}) =& \frac{\rho}{12}\left(\sqrt{3}\pi\left(\rho^2-16\beta\right)+\left(e^{2\sqrt{3}\pi}-1\right)w\left(\rho+e^{2\sqrt{3}\pi}w+w\right)\right),\\
g_{12}(\mathbf{z}) =& \frac{1}{144}\left(\rho^3-e^{2\sqrt{3}\pi}\left(\rho^3+12\sqrt{3}\pi\rho^2w+28w^3+192\sqrt{3}\pi\beta w\right)+28e^{6\sqrt{3}\pi}w^3\right),\\
g_{21}(\mathbf{z}) =& \frac{\left(1+16\sqrt{3}\pi+54\pi^2\right)\rho^5}{1728}+\frac{e^{14\sqrt{3}\pi}\rho w^4}{108}+\frac{e^{10\sqrt{3}\pi}\rho w^3(15\rho-196w)}{15120}\\
&+\frac{e^{12\sqrt{3}\pi}\rho w^3(21\rho+13w)}{5616}+\frac{e^{8\sqrt{3}\pi}\rho w^2\left(171\rho^2-700\rho w+3192\left(w^2-\beta\right)\right)}{229824}\\
&+\frac{e^{2\sqrt{3}\pi}\left(\rho^2w\left(288\beta+48\sqrt{3}\pi\left(\rho^2-16\beta\right)-19\rho^2\right)-2\rho^5\right)}{3456}+\frac{467\rho^4w}{169344}\\
&-\frac{5\rho^2\left(3815w^3+28652\beta w\right)}{1742832}-\frac{\rho^3\left(3024\pi\left(\sqrt{3}+4\pi\right)\beta+115w^2\right)}{18144}\\
&+\frac{e^{6\sqrt{3}\pi}\rho w^2\left(112\beta-42\sqrt{3}\pi\left(16\beta+\rho^2\right)+31\rho^2+42\rho w\right)}{4536}\\
&-\frac{e^{4\sqrt{3}\pi}\rho w}{84672}\left(-232\rho^3+105\rho^2w+84\sqrt{3}\pi\left(5\rho^3-7\rho^2w+80\beta\rho+112\beta w\right)\right.\\
&\left.+96\beta\rho+9408\beta w\right)+\rho\left(\frac{4}{3}\pi\left(2\pi\beta^2-\sqrt{3}\beta_2\right)-\frac{w^4}{80}+\frac{65\beta w^2}{648}\right),\\
g_{22}(\mathbf{z}) =& e^{10\sqrt{3}\pi}\left(-\frac{\rho^2w^3}{252}-\frac{7\rho w^4}{456}+\frac{19w^5}{480}\right)+\frac{\rho^3\left(32\beta+12\sqrt{3}\pi\left(\rho^2-16\beta\right)-\rho^2\right)}{6912}\\
&+\left(\rho^2\left(1071w-\left(619+504\sqrt{3}\pi\right)\rho\right)-288\beta\left(\left(6+28\sqrt{3}\pi\right)\rho+49w\right)\right)\\
&\frac{e^{6\sqrt{3}\pi}w^2}{84672}+\frac{e^{8\sqrt{3}\pi}w^2}{24192}\left(\rho\left(-21\rho^2+4\left(73-196\sqrt{3}\pi\right)\rho w+644w^2\right)\right.\\
&\left.+112\beta\left(3\rho+28\left(1-4\sqrt{3}\pi\right)w\right)\right)+\frac{e^{4\sqrt{3}\pi}w}{22464}\left(-9984\beta^2-123\rho^4\right.\\
&\left.+377\rho^3w-1248\beta\rho^2+78\sqrt{3}\pi\left(16\beta+\rho^2\right)^2-936\beta\rho w\right)+\frac{7e^{16\sqrt{3}\pi}w^5}{216}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{e^{12\sqrt{3}\pi} w^3 \left(49\rho^2 + 225\rho w + 196\left(4\beta + 7w^2\right)\right)}{30240} + \frac{35 e^{14\sqrt{3}\pi} \rho w^4}{2808} \\
&+ e^{2\sqrt{3}\pi} \left(\frac{\left(41 - 52\pi\left(\sqrt{3} + 3\pi\right)\right)\rho^4 w}{7488} - \frac{\rho^3\left(784\left(1 - 2\sqrt{3}\pi\right)\beta + 1457w^2\right)}{169344} \right. \\
&- \frac{\rho^2 w\left(3360\left(3\left(\sqrt{3} - 2\pi\right)\pi - 1\right)\beta + 1157w^2\right)}{60480} + \rho\left(\frac{85\beta w^2}{1764} - \frac{6085 w^4}{373464}\right) \\
&\left. - \frac{23 w^5}{864} + \frac{\left(1 + 4\sqrt{3}\pi\right)\rho^5}{6912} + \frac{17\beta w^3}{270} + \frac{4}{9} w\left(\beta^2 - 3\sqrt{3}\pi\beta_2\right) \right).
\end{aligned}
$$

We point out that the function $\mathbf{g}_0(\mathbf{z})$ satisfies the hypothesis (i) for the graph $\mathcal{Z} = \{(\alpha, 0) : \alpha > 0\}$. We apply Theorem 1 to system (24) taking $r = 0$. Then we have $\Delta_\alpha = 1 - e^{-2\sqrt{3}\pi} \neq 0$ and the function

$$
f_1(\alpha) = \frac{\pi\alpha\left(\alpha^2 - 16\beta_1\right)}{4\sqrt{3}},
$$

has the positive simple zero $\alpha^* = 4\sqrt{\beta_1}$, where $Df_1(\alpha^*) = 8\pi\beta_1/\sqrt{3}$. Then system (24) has a 2π-periodic orbit by Theorem 1. The periodic orbit of system (10) is obtained going back through the change of variables. Now we want to study the stability of this periodic orbit using Theorem 2. First using (4) we compute the function

$$
\begin{aligned}
f_2(\alpha) = &\frac{\alpha}{1728}\Big(2304\pi\left(2\pi\beta^2 - \sqrt{3}\beta_2\right) + \left(1 - 2e^{2\sqrt{3}\pi} + e^{4\sqrt{3}\pi} + 16\sqrt{3}\pi + 54\pi^2\right)\alpha^4 \\
&- 288\pi\left(\sqrt{3} + 4\pi\right)\beta\alpha^2\Big).
\end{aligned}
$$

Then if $\varphi(t, \varepsilon)$ is the periodic solution founded above we can use (21) and (8) to write $\varphi(0, \varepsilon) = \mathbf{z}_0 + \varepsilon\mathbf{z}_1$ where

$$
\begin{aligned}
\mathbf{z}_0 &= \left(4\sqrt{\beta_1}, 0\right), \\
\mathbf{z}_1 &= \left(-\frac{2\left(\left(1 - 2e^{2\sqrt{3}\pi} + e^{4\sqrt{3}\pi} - 2\sqrt{3}\pi\right)\beta^2 - 9\sqrt{3}\pi\beta_2\right)}{9\sqrt{3}\pi\sqrt{\beta}}, \right. \\
&\left. \quad - \frac{4e^{2\sqrt{3}\pi}\beta^{5/2}}{27}\left(4\sqrt{3}\pi\left(1 + \coth\left(\sqrt{3}\pi\right)\right) - 1\right) \right).
\end{aligned}
$$

Then by (6) and (7) we can write the matrix (5) as

$$
A(\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^{-2\sqrt{3}\pi} \end{pmatrix} + \varepsilon \begin{pmatrix} \dfrac{8\pi\beta_1}{\sqrt{3}} & \dfrac{4\beta_1}{3}\left(e^{2\sqrt{3}\pi} - 1\right) \\ -\dfrac{\beta_1}{3}\left(e^{2\sqrt{3}\pi} - 1\right) & -\dfrac{8e^{2\sqrt{3}\pi}\pi\beta_1}{\sqrt{3}} \end{pmatrix}.
$$

The matrix $A(\varepsilon)$ has two eigenvalues $\lambda_1 = \varepsilon\dfrac{8\pi\beta_1}{\sqrt{3}} + \mathcal{O}(\varepsilon^2)$ and $\lambda_2 = 1 - e^{-2\sqrt{3}\pi} - \varepsilon\dfrac{8e^{2\sqrt{3}\pi}\pi\beta_1}{\sqrt{3}} + \mathcal{O}(\varepsilon^2)$. Consequently this matrix satisfies the hypothesis (s_1) and (s_2) with $c = \beta_1$, i. e., $|\lambda_i| > \varepsilon\beta_1$ for $i = 1, 2$ and $\varepsilon > 0$ sufficiently small. □

Proof of Theorem 5. We will prove the result only for the equilibrium point $\mathbf{P}_+$. The proof for the point $\mathbf{P}_-$ follows exactly the same steps. First we translate the equilibrium point $\mathbf{P}_+$ to the origin and rescale the system using the change of variables $(X,Y,Z) = \sqrt{\varepsilon}\big(x+z,(-x-\sqrt{3}y+2z)/2,(-x-\sqrt{3}y+2z)/2\big)$ the differential system (11) becomes

$$
\begin{aligned}
\dot{X} =& -\omega Y+\sqrt{\varepsilon}\Big(X^2+2X\left(\sqrt{3}Y+2Z\right)-Y\left(Y+4\sqrt{3}Z\right)\Big)\frac{3\omega\sqrt[4]{3}}{4\sqrt{2\omega}}\\
&+\frac{\varepsilon}{8}\Big(X\left(8a_1-20b_1+3\left(Y^2+4\sqrt{3}YZ+4Z^2\right)\right)+Y\left(-3\sqrt{3}Y^2-6YZ\right.\\
&\left.-4\sqrt{3}\left(2a_1-3b_1+3Z^2\right)\right)+3X^3-3X^2\left(\sqrt{3}Y-2Z\right)\Big)+\mathcal{O}(\varepsilon^{3/2}),\\
\dot{Y} =&\omega X+\sqrt{\varepsilon}(\sqrt{3}X^2-2XY+4\sqrt{3}XZ-\sqrt{3}Y^2+4YZ)\frac{3\omega\sqrt[4]{3}}{4\sqrt{2\omega}}\\
&+\frac{\varepsilon}{8}\Big(8a_1\left(\sqrt{3}X+Y\right)-4b_1\left(3\sqrt{3}X+5Y\right)+3\Big(\sqrt{3}X^3+X^2\left(Y+2\sqrt{3}Z\right)\\
&+X\left(\sqrt{3}Y^2-4YZ+4\sqrt{3}Z^2\right)+Y\left(Y^2-2\sqrt{3}YZ+4Z^2\right)\Big)\Big)+\mathcal{O}(\varepsilon^{3/2}),\\
\dot{Z} =&-\sqrt{3}\omega Z+\sqrt{\varepsilon}(X^2+Y^2+2Z^2)\frac{3\sqrt{\omega}\sqrt[4]{3}}{2\sqrt{2}}+\frac{\varepsilon}{4}\Big(8Z(b_1-a_1)-X^3\\
&-6Z\left(X^2+Y^2\right)+3XY^2-4Z^3\Big)+\mathcal{O}(\varepsilon^{3/2}).
\end{aligned}
$$

This system can be written into the normal form for applying the averaging theory. We use the cylindrical change of variables $(X,Y,Z) = \big(\rho\cos\theta,\rho\sin\theta,w\big)$ with $\rho>0$. Then we check that $\dot{\theta} = \sqrt{3}/2+\mathcal{O}(\varepsilon^{1/2})$ for $\varepsilon>0$ sufficiently small. Then we take θ as the new independent variable obtaining the differential system

$$
\dot{\mathbf{z}} = \mathbf{F}_0\big(\mathbf{z},\theta\big)+\sqrt{\varepsilon}\mathbf{F}_1\big(\mathbf{z},\theta\big)+\varepsilon\mathbf{F}_2\big(\mathbf{z},\theta\big)+\mathcal{O}(\varepsilon^{3/2}), \tag{25}
$$

with $\mathbf{z}=(\rho,w)$, $\mathbf{F}_0\big(\mathbf{z},\theta\big) = \big(0,-\sqrt{3}w\big)$, and $\mathbf{F}_i\big(\mathbf{z},\theta\big) = \big(F_{i1}\big(\mathbf{z},\theta\big),F_{i2}\big(\mathbf{z},\theta\big)\big)$ for $i=1,2$, where

$$
\begin{aligned}
F_{11}\big(\mathbf{z},\theta\big) =&\frac{3\sqrt[4]{3}\rho\left(\sqrt{3}\rho\sin(3\theta)+\rho\cos(3\theta)+4w\right)}{4\sqrt{2}\sqrt{\omega}},\\
F_{12}\big(\mathbf{z},\theta\big) =&-\frac{3\sqrt[4]{3}\left(2\rho^2-8w^2+\sqrt{3}\rho w\sin(3\theta)-3\rho w\cos(3\theta)\right)}{4\sqrt{2}\sqrt{\omega}},\\
F_{21}\big(\mathbf{z},\theta\big) =&-\frac{\rho}{32\omega}\Big(3\rho\left(9\rho\cos(6\theta)+2\sqrt{3}\sin(3\theta)(3\rho\cos(3\theta)+8w)+64w\cos(3\theta)\right)\\
&+4\left(-8a_1+20b_1-3\rho^2+96w^2\right)\Big),\\
F_{22}\big(\mathbf{z},\theta\big) =&\frac{1}{32\omega}\Big(\rho\left(6\sqrt{3}\sin(3\theta)\left(-3\rho^2+26w^2+9\rho w\cos(3\theta)\right)+\left(46\rho^2-468w^2\right)\cdot\right.\\
&\left.\cos(3\theta)-27\rho w\cos(6\theta)\right)+2w\left(16a_1-40b_1+75\rho^2-376w^2\right)\Big).
\end{aligned}
$$

We consider the period $T=2\pi$, then system (25) is into normal form for applying Theorem 1. Taking the initial condition $\mathbf{z}_0 = (\rho_0,w_0)$ the solution of the unperturbed differential system corresponding to (25) is given by $\Phi(\theta,\mathbf{z}) = \Big(\rho_0,w_0e^{-\sqrt{3}\theta}\Big)$. Again we consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. Thus for $\mathbf{z}_\alpha\in\mathcal{Z}$ the solution $\Phi(\theta,\mathbf{z}_\alpha)$ is 2π-periodic, and therefore the differential system (25) satisfies the hypothesis (H). Moreover the fundamental matrix of the variational differential system along $\Phi(\theta,\mathbf{z}_\alpha)$ is

$$
M(\theta,\mathbf{z}_\alpha) = \begin{pmatrix}1 & 0\\ 0 & e^{-\sqrt{3}\theta}\theta\end{pmatrix}.
$$

The averaging functions for this system are $\mathbf{g}_0(\mathbf{z}) = \left(0, (1-e^{2\pi\sqrt{3}})w\right)$ and $\mathbf{g}_i(\mathbf{z}) = (g_{i1}(\mathbf{z}), g_{i2}(\mathbf{z}))$ for $i = 1, 2$ where

$$
\begin{aligned}
g_{11}(\mathbf{z}) =& \frac{3^{3/4}\left(1-e^{-2\sqrt{3}\pi}\right)\rho w}{\sqrt{2}\sqrt{\omega}},\\
g_{12}(\mathbf{z}) =& -\frac{3^{3/4}e^{-4\sqrt{3}\pi}\left(e^{2\sqrt{3}\pi}-1\right)\left(e^{2\sqrt{3}\pi}\rho^2-4w^2\right)}{2\sqrt{2}\sqrt{\omega}},\\
g_{21}(\mathbf{z}) =& \frac{\rho e^{-8\sqrt{3}\pi}}{112\omega}\Big(e^{8\sqrt{3}\pi}\left(28\pi\left(8a_1-5\left(4b_1+3\rho^2\right)\right)+\sqrt{3}\left(84\rho^2-168w^2-23\rho w\right)\right)\\
&-56\sqrt{3}e^{2\sqrt{3}\pi}w^2+84\sqrt{3}w^2+\sqrt{3}e^{4\sqrt{3}\pi}w(51\rho+140w)\\
&-28\sqrt{3}e^{6\sqrt{3}\pi}\rho(3\rho+w)\Big),\\
g_{22}(\mathbf{z}) =& \frac{e^{-10\sqrt{3}\pi}}{8736\omega}\Big(-1820\sqrt{3}e^{10\sqrt{3}\pi}\rho^3+26208\sqrt{3}w^3+1092\sqrt{3}e^{2\sqrt{3}\pi}w^2(3\rho-32w)\\
&-52\sqrt{3}e^{4\sqrt{3}\pi}w^2(81\rho-658w)-39879\sqrt{3}e^{6\sqrt{3}\pi}\rho^2w+e^{8\sqrt{3}\pi}\big(2184\pi w.\\
&\left(8a_1-20b_1-75\rho^2\right)+\sqrt{3}\left(1820\rho^3-25480w^3+936\rho w^2+39879\rho^2 w\right)\big)\Big).
\end{aligned}
$$

Function $\mathbf{g}_0(\mathbf{z})$ vanishes on the the graph $\mathcal{Z} = \{(\alpha, 0) : \alpha > 0\}$. We apply Theorem 1 to system (25). Here $r = 0$ and $\Delta_\alpha = 1 - e^{-2\sqrt{3}\pi} \neq 0$. The bifurcation functions are

$$
\begin{aligned}
f_1(\alpha) &= 0,\\
f_2(\alpha) &= \frac{3\left(\sqrt{3}e^{-4\sqrt{3}\pi}\left(1-2e^{2\sqrt{3}\pi}\right)+\sqrt{3}-5\pi\right)\alpha^3+8\pi\alpha a_1-20\pi\alpha b_1}{4\omega}.
\end{aligned}
$$

Function f_2 has the positive simple zero

$$
\alpha^* = 2e^{2\sqrt{3}\pi}\sqrt{\frac{\pi(5b_1-2a_1)}{3\sqrt{3}-6\sqrt{3}e^{2\sqrt{3}\pi}+3\sqrt{3}e^{4\sqrt{3}\pi}-15e^{4\sqrt{3}\pi}\pi}},
$$

where $Df_2(\alpha^*) = (10\pi b_1 - 4\pi a_1)/\omega$. By statement (b) of Theorem 1 system (25) has a 2π-periodic solution. The periodic solution of system (12) is obtained going back through the change of variables. □

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