ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.25(2018) No.1,pp.3-26

Zero–Hopf Bifurcations in A Hyperchaotic Lorenz System II

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(Received 12 September 2017, accepted 20 December 2017)

Abstract: Recently sixteen 3-dimensional differential systems exhibiting chaotic motion and having no equilibria have been studied, and it has been graphically observed that these systems have a period-doubling cascade of periodic orbits providing the route to their chaotic motions. Here using new results on the averaging theory we prove that these systems exhibit, for some values of their parameters different to the ones having chaotic motion, either a zero–Hopf or a Hopf bifurcation, and graphically we observed that the periodic orbit starting in those bifurcations is at the beginning of the mentioned period–doubling cascade.

Keywords: hyperchaotic Lorenz system; zero–Hopf bifurcation; periodic orbits; averaging theory

1 Introduction

The Lorenz system of differential equations in $\mathbb{R}^3$ arose from the work of meteorologist/mathematician Edward N. Lorenz [1], who studied forced dissipative hydrodynamical systems. As he was computing numerical solutions to the system, he notice that initial conditions with small differences eventually produced vastly different solutions, a characteristic of chaos. Since that time, about 1963, the Lorenz system has become one of the most widely studied systems of ODEs because of its wide range of behaviors. Although the origins of this system lies in atmospheric modeling, the Lorenz equations also appear in other areas as in the modeling of lasers see [2], and dynamos see [3].

In recent times a so-called *hyperchaotic Lorenz* system was introduced; see for instance [4–16] and the references therein. MathSciNet presently lists 32 papers on *hyperchaotic Lorenz* systems. We observe that not all these hyperchaotic Lorenz systems are similar, since they can vary in one or two terms. However these systems can be precisely definite as autonomous differential systems in a phase space of dimension at least four, with a dissipative structure, and at least two unstable directions, such that at least one is due to a nonlinearity. The hyperchaotic systems has a dynamics hard to predict or control, for this reason such systems are as well of use in secure communications systems see, for instance [17].

Our aim in this work is to study, from a dynamical point of view, the 4*–dimensional zero–Hopf equilibria* in the hyperchaotic Lorenz system. Here, a 4–dimensional zero–Hopf equilibrium means an equilibrium point with two zeros and a pair of pure conjugate imaginary numbers as eigenvalues. Using the method of averaging and convenient changes of variables and parameters we can analyse the zero–Hopf bifurcations. More precisely we study zero–Hopf bifurcations of the following hyperchaotic Lorenz system (as given in [6, 9])

$$\begin{aligned} \dot{x} &= a(y-x)+w, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= -bz+xy, \\ \dot{w} &= dw-xz, \end{aligned} \tag{1}$$

for appropriate choices of the parameters a, b, c and d.

There are several works studying zero–Hopf bifurcation see for instance Guckenheimer [18], Guckenheimer and Holmes [19], Han [20], Kuznetsov [21], Llibre [22], Marsden, Scheurle [23].... It has been shown that, under specific conditions, some elaborated invariant sets of the unfolding could be bifurcated from a zero–Hopf equilibrium and hence,

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IJNS.2018.02.15/990