NON-EXISTENCE, EXISTENCE, AND UNIQUENESS OF LIMIT CYCLES FOR A GENERALIZATION OF THE VAN DER POL–DUFFING AND THE RAYLEIGH–DUFFING OSCILLATORS

MURILO R. CÂNDIDO[1], JAUME LLIBRE[2] AND CLAUDIA VALLS[3]

Abstract. We study the limit cycles of some cubic family of differential equations, containing the well-known van der Pol-Duffing and Rayleigh-Duffing oscillators. In particular, we characterize for the class of differential systems here studied the non-existence, existence and uniqueness of limit cycles. Moreover we provide their global phase portraits in the Poincaré disc.

1. Introduction and statement of the main results

In this paper we consider the planar system

$$x' = y, \quad y' = -a_1x - a_2x^3 + \mu(a_3 + a_4x^2 + a_5y^2)y, \tag{1}$$

where $a_1, a_2 > 0$, $a_3, a_4, a_5 \in \mathbb{R}$, $a_3 \neq 0$, $\mu > 0$ is a sufficiently small parameter, $a_5 < 0$ and the prime denotes derivative with respect to the time t.

The Van der Pol oscillator was discovered by engineer and physicist Balthasar Van der Pol while working at Philips company. Van der Pol [26] found in circuits that employ vacuum valves stable oscillations, which are now known as limit cycles. Van der Pol also found that at certain frequencies, some irregular noise appears near the coupling frequencies. It will be one of the first experimental discoveries of Chaos Theory [13, 27]. The Van der Pol equation has a long history not only in physics but also in biology. Thus in biology Fitzhugh [8] and Nagumo [22] applied the equation to a two-dimensional field in the Fitzhugh-Nagumo model, as it is known now, to describe the potential action of neurons. It can also be used in seismology to model the behavior of

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05, 37C10.

Key words and phrases. limit cycles, van der Pol-Duffing oscillator, Rayleigh-Duffing oscillator.

plates in a failure [1]. We note that the Rayleigh equation introduced by Rayleigh in 1875 in his published book *The Theory of Sound* [24], where he shown the first physical phenomenon modeled by a limit cycle. In fact the Rayleigh equation is more general than the Van der Pol equation. The Duffing equation introduced in [3] essentially adds a term x^3 to the Rayleigh equation.

The differential systems (1) contain as particular cases the so-called Van der Pol-Duffing oscillator and Rayleigh-Duffing oscillator both with positive linear damping and sufficiently small stiffness. Both models have been studied intensively because they are two essential oscillators in the nonlinear dynamical systems, see for instance the books of [22, 23] and the references cited therein. Many researchers have investigated the existence of limit cycles for autonomous nonlinear systems depending on parameters and in special for system (1) due to the fact that they can be a mechanism for the creation of chaos, see [11, 19]. This problem is also related with the well-known 16th Hilbert problem which asks for the number of limit cycles in polynomial differential equations in function of their degree, see for instance [9, 10, 15].

The main objective of this paper is to characterize the dynamics of the differential system (3) in an easy way using the qualitative theory of differential equations, and in particular we provide the non-existence, existence and uniqueness of limit cycles for the differnetial equation (3). Consequently we provide a new unified proof in the study of the limit cycles of the Van der Pol-Duffing equation and of the Rayleigh-Duffing equation.

The differential equation (1) when $a_5 = 0$ becomes a subclass of the classical polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

introduced in [17], when we write it as the differential system of first order

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y.$$

Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively. This differential equation has been studied by many authors. We denote by $H(m, n)$ the maximum number of limit cycles that the differential equation (2) with m and n fixed can have. Now we describe briefly some of the main results on the limit cicles on the Liénard differential equation (2)

(i) In 1928 Liénard [17] showed if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (2) has a unique limit cycle.
(ii) In 1973 Rychkov [25] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (2) has at most two limit cycles.
(iii) In 1977 Lins, de Melo and Pugh [18] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.
(iv) In 1998 Coppel [2] proved that $H(2, 1) = 1$.
(v) Dumortier, Li and Rousseau in [7] and [4] proved that $H(3, 1) = 1$.
(vi) In 1997 Dumortier and Chengzhi [5] proved that $H(2, 2) = 1$.

For the differential system (1), with $\mu = 1$, $a_1 = -1$, $a_2 = 1$, $a_4 = 0$ and $a_5 = -1$, the authors in [12] studied numerically the creation and annihilation of limit cycles depending on the negative parameter a_3 (that is for negative linear damping and negative linear stiffness). In particular in [12] nothing is studied when a_1 is positive and for other the values of the parameters a_k for $k = 2, ..., 5$.

Note that since $a_1, a_2 > 0$, by the change of coordinates and a reparametrization of time of the form

$$x = \sqrt{\frac{a_1}{a_2}}X, \quad y = \frac{a_1}{\sqrt{a_2}}Y, \quad t = \frac{1}{\sqrt{a_1}}\tau,$$

we can write system (1) as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - x^3 - \varepsilon(by + x^2y + ay^3), \end{aligned} \tag{3}$$

where $\varepsilon = -a_4\sqrt{a_1}\mu/a_2$, $b = a_2a_3/(a_4a_1)$ and $a = a_5a_1/a_4$. The dot denotes derivative with respect to the new variable τ and we have renamed the new variables (X, Y) as (x, y). System (3) is invariant by the change $(x, y, \varepsilon, t) \mapsto (-x, y, -\varepsilon, -t)$, and so from now on and without loss of generality we can assume that $\varepsilon > 0$, i.e. $a_4 \leq 0$. Since $a_5 < 0$ we have that $a > 0$. So in the rest of this paper and without loss of generality we assume that $\varepsilon > 0$ and $a > 0$ in the differential system (3).

Note that system (3) with $\varepsilon = 0$ has a first integral of the form

$$H = \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^4}{4}. \tag{4}$$

To state the main result of this paper we recall that a *limit cycle* is a periodic orbit which is isolated in the set of all periodic orbits of the system. In general the limit cycles are very difficult to detect. The main result of our paper is the following.

Theorem 1. *The polynomial differential system* (3) *has no limit cycles when* $b \geq 0$*.*

Theorem 1 is proved in section 3.

Theorem 2. *The differential system* (3) *has a unique stable limit cycle when* $b < 0$*.*

See Chapter 5 of [6] for the definitions of Poincaré disc and Poincaré compactification of a polynomial differential system, and Chapter 1 of [6] for the definition of topological equivalence. The second main result of the paper is the following.

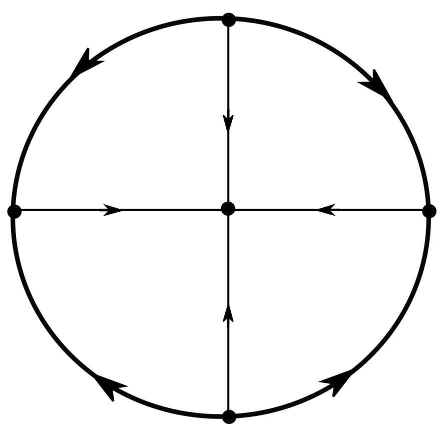

Figure 1

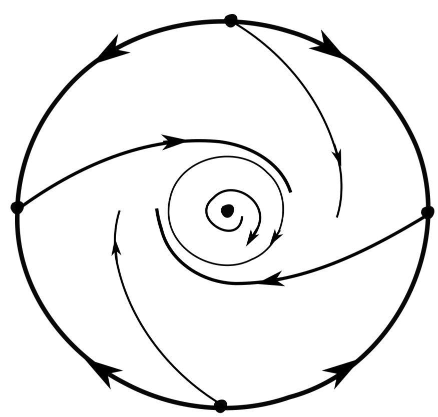

Figure 2

Theorem 3. *The global phase portrait of the differential system* (3) *in the Poincaré disc is topologically equivalent to the one of Figure* 1 *if* $b \geq 0$ *and to the one of Figure* 2 *if* $b < 0$*.*

Theorems 2 and 3 are proved in section 4.

2. Preliminary results and definitions

If a planar differential system has some orbit such that its ω–limit escapes at infinity is called an *unbounded system*, otherwise it is called a *bounded system*. Note that system (3) is bounded.

Suppose that as α varies in (a, b), the equililbrium points of the vector fields $(P(x, y, \alpha), Q(x, y, \alpha))$ remain unchanged, and for any given point $p = (x, y)$ and any parameters $\alpha_1 < \alpha_2$ in (a, b), we have

$$\begin{vmatrix} P(x,y,\alpha_1) & Q(x,y,\alpha_1) \\ P(x,y,\alpha_2) & Q(x,y,\alpha_2) \end{vmatrix} \geq 0 \text{ (or } \leq 0), \tag{5}$$

where equality cannot hold on an entire periodic orbit of the vector field $(P(x, y, \alpha_i),\ Q(x, y, \alpha_i)),\ i = 1, 2$. Then the vector fields $(P(x, y, \alpha), Q(x, y, \alpha))$ are called *generalized rotated vector fields*. Here, the interval (a, b) can be either bounded or unbounded.

The generalized rotated vector fields have the following two properties.

- (i) The stable (respectively unstable) periodic orbits which are run in clockwise sense increase (respectively decrease) their size as the parameter increases.
- (ii) The periodic orbits of the system for different values of the parameter cannot intersect.

For the proofs of these properties see for instance [6, 29].

3. Proof of Theorem 1

We recall that a *center* p of a differential system defined in $\mathbb{R}^2$ is a equilibrium point of the system for which it exists a neighborhood U such that $U \setminus \{p\}$ is filled of periodic orbits. Moreover the center p is *global* is $\mathbb{R}^2 \setminus \{p\}$ is filled of periodic orbits.

Lemma 4. *The differential system* (3) *with* $\varepsilon = 0$ *has a global center at the origin of coordinates.*

Proof. The differential system (3) with $\varepsilon = 0$ is a Hamiltonian system with Hamiltonian H given in (4). Since all the level curves of $H = h$ with $h > 0$ are closed curves surrounding the origin of coordinates, which leaves at $H = 0$, the origin is a global center. □

Proof of Theorem 1. Let $(x(t), y(t))$ be an arbitrary solution of the polynomial differential system (3) different from the equilibrium point $(0, 0)$. Since the function $H = H(x, y)$ given in (4) evaluated on the solutions of the differential system (3) satisfies

$$\frac{dH}{dt}(x(t), y(t)) = \frac{\partial H}{\partial x}\dot{x}(t) + \frac{\partial H}{\partial y}\dot{y}(t) = -\varepsilon y(t)^2(b + x(t)^2 + ay(t)^2) \leq 0,$$

because $a > 0$, $\varepsilon > 0$, and by assumption $b \geq 0$ is also positive, it follows that on orbit $(x(t), y(t))$ of the differential system (3) the value of the function H decreases when the time increases. Therefore the orbit $(x(t), y(t))$ tends to the origin of coordinates, the equilibrium point of the Hamiltonian system (3) with $\varepsilon = 0$, hence the origin is a global attractor for system (3). In other words the function H is a Lyapunov function for system (3) in the whole $\mathbb{R}^2$, and consequently the origin is a global attractor, for more details see Theorem 1.35 of [6].

Since the origin is a global attractor system (3) with $b \geq 0$ has no periodic orbits, and so no limit cycles. This completes the proof of Theorem 1. □

4. Proofs of Theorems 2 and 3

Proof of Theorem 2. We first study the finite equilibrium points of system (3). Note that the unique finite equilibrium point is the origin. The Jacobian matrix of system (3) at the origin is

$$M = \begin{pmatrix} 0 & 1 \\ -1 & -b\varepsilon \end{pmatrix}.$$

whose eigenvalues are

$$\frac{-b\varepsilon \pm \sqrt{(b\varepsilon)^2 - 4}}{2}.$$

Hence, if $b = 0$ the origin is a weak focus because the first non–zero Liapunov constant is $-\varepsilon(3a + 1)\pi/4$. Since $\varepsilon > 0$ it is stable weak focus. Moreover, if $b \neq 0$ then if $|b\varepsilon| \geq 2$ the origin is a node, and if $|b\varepsilon| < 2$ it is a focus. It is stable if $b > 0$, and unstable if $b < 0$.

Now we study the infinite equilibrium points. On the local chart U_1 system (3) becomes

$$\begin{aligned} \dot{u} &= -1 - v^2 - u^2v^2 - \varepsilon u(1 + au^2 + bv^2), \\ \dot{v} &= -uv^3. \end{aligned}$$

Note that this system has the infinite equilibrium point

$$\left(\frac{2\sqrt[3]{3}a\varepsilon^2 - \sqrt[3]{2}\left(\sqrt{3}a^{3/2}\varepsilon^2\sqrt{27a + 4\varepsilon^2} + 9a^2\varepsilon^2\right)^{2/3}}{6^{2/3}a\varepsilon(\sqrt{3}a^{3/2}\varepsilon^2\sqrt{27a + 4\varepsilon^2} + 9a^2\varepsilon^2)^{1/3}}, 0 \right).$$

The eigenvalues of the Jacobian matrix at that point are 0 and

$$\varepsilon - \frac{6\ 2^{2/3}\sqrt[3]{3}a\varepsilon^3}{\left(a^{3/2}\varepsilon^2\left(\sqrt{81a+12\varepsilon^2}+9\sqrt{a}\right)\right)^{2/3}} - \frac{\sqrt[3]{2}3^{2/3}\left(a^{3/2}\varepsilon^2\left(\sqrt{81a+12\varepsilon^2}+9\sqrt{a}\right)\right)^{2/3}}{a\varepsilon}.$$

So it is semi-hyperbolic,and applying Theorem 2.19 in [6] we get that it is a semi-hyperbolic saddle.

On the local chart U_2 system (3) becomes

$$\dot{u} = v^2 + u^4 + u^2v^2 + \varepsilon u(a + u^2 + bv^2),$$
$$\dot{v} = v(u^3 + uv^2 + \varepsilon(a + u^2 + bv^2)).$$

So the origin of U_2 is an equilibrium point whose eigenvalues of the Jacobian matrix at the origin are both equal to εa. Hence, the origin of the local chart U_2 is an unstable node because $\varepsilon a > 0$.

Now we shall prove that for $b < 0$ the differential system (3) has a unique limit cycle.

Since the origin is the unique finite equilibrium point which is unstable for $b < 0$ and the system is bounded, by the Poincaré-Bendixson Theorem (see for instance Theorem 1.25 of [6]) it must exists at least one stable limit cycle. This limit cycle borns in a Hopf bifurcation at $b = 0$ because for $b \in (-2, 0])$ the origin is an unstable focus, and for $b \in (0, 2)$ the origin is a stable focus, for more details on the Hopf bifurcation see [14].

The vector fields associated to our system (3) are generalized rotated vector fields with respect to the parameter b in the whole $\mathbb{R}$, because the determinant (5) for this system is $\varepsilon(b_1 - b_2)y^2$, and the system has the origin as the unique equilibrium point which does not depend on the parameter b.

Suppose that system (3) has more than one limit cycle. The closest limit cycle to the origin, γ_1, must be stable because the origin for $b < 0$ is unstable. The limit cycle γ_2 nearest to γ_1 must be unstable in the bounded region limited by it. In view of the properties (i) and (ii) given in section 2 we get a contradiction, because increasing the parameter b the limit cycle γ_1 increases and the limit cycle γ_2 decreases, so for some value of b they would intersect. □

Proof of Theorem 3. We recall that when $b \geq 0$, by Theorem 1, the differential system (3) has no limit cycles, and that by Theorem 2 system (3) has a unique stable limit cycle when $b < 0$. Moreover in the proof of Theorem 2 we have seen:

(i) The differential system (3) is bounded for all values of the parameter b.
(ii) The endpoints of the x-axis at infinity are saddles, and the endpoints of the y-axis at infinity are unstable nodes.
(iii) The unique finite equilibrium point, the origin, is stable when $b \geq 0$ and unstable when $b < 0$.

Taking into account all this information on the local phase portraits at the finite and infinite equilibrium points, and the non-existence, existence and uniqueness of the limit cycles, the phase portraits in the Poincaré disc of system (3) are topologically equivalent to the one of Figure 1 if $b \geq 0$, or to the one of Figure 2 if $b < 0$. □

Acknowledgements

M. Cândido is partially supported by FAPESP 2018/07344-0 and 2019/05657-4. J. Llibre is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. C. Valls is partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] J. Cartwright, V. Eguiluz, E. Hernández-Garcia and O. Piro, *Dynamics of elastic excitable mitjan*, Internat. J. Bifur. Chaos **9** (1999), 2197–2202.
[2] W. A. Coppel, *Some quadratic systems with at most one limit cycles*, Dynamics Reported, Vol.**2**,Wiley, New York, 1998, pp. 61–68.
[3] G. Duffing, *Forced oscillations with variable natural frequency and their technical relevance*, (in German), Heft 41/42, Braunschweig: Vieweg, (1918).
[4] F. Dumortier and C. Li, *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1500.
[5] F. Dumortier and C. Li, *Quadratic Liénard equations with quadratic damping*, J. Diff. Eqs. **139** (1997), 41–59.
[6] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer Verlag, New York, 2006.
[7] F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear dapimg*, Nonlinearity **3** (1990), 1015–1039.
[8] R. Fitzhugh, *Impuls and physiological states in theoretical models of nerve membranes*, Biophysics J. **1** (1961), 445–466.

[9] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ottingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.

[10] Yu. Ilyashenko, *Centennial history of Hilbert's* 16*th problem*, *Bull. (New Series) Amer. Math. Soc.* **39** (2002), 301–354.

[11] S. Jafari, J. C. Sprott, S. Hashemi Golpayegani and S. Mohammad Reza, *Elementary quadratic chaotic flows with no equilibria*, Physics Letters A **377** (2013), 699–702.

[12] Y. Kanai and H. Yabuno, *Creation-annihilation process of limit cycles in the Rayleigh-Duffing oscillator*, Nonlinear Dyn. **70** (2012), 1007–1016.

[13] T. Kanamaru, *Van der Pol oscillator*, Scholarpedia **2** (1) (2007), 2202.

[14] Y.A. Kuznetsov, *Elements of applied bifurcation theory*, Second Edition, Applied Mathematical Sciences, Vol. **112**, Springer, 1998.

[15] J. Li, *Hilbert's* 16*th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.

[16] X. Li, J.C. Ji, C.H. Hansen and C. Tan, *The response of a a Duffing-van der Pol oscillator under delayed feedback control*, J. Sound. Vib. **291** (2006), 644–655.

[17] A. Liénard, *Étude des oscillations entrenues*, Revue Génerale de l' Électricité **23** (1928), 946–954.

[18] A. Lins, W. de Melo and C.C. Pugh, *On Liénard's Equation*, Lecture Notes in Math **597**, Springer, Berlin, (1977), pp. 335–357.

[19] J. Llibre, M. Messias and A. C. Reinol, *Zero-Hopf bifurcations in three-dimensional chaotic systems with one stable equilibrium*, preprint, (2019).

[20] K. Moukam, S. Bowong, C. Tchawoua and E. Kaptouom, *Strange attractors and chaos control in a Duffing-van der Pol oscillator with two external periodic forces*, J. Sound. Vib. **277** (2004), 783.

[21] J. Nagumo, S. Arimoto and S. Yoshizawa, *An activeu premeu transmission line simulating nerve axon*, Proc. IRE **50** (1962), 2061–2070.

[22] A.H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1981.

[23] A.H. Nayfeh and D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.

[24] L. Rayleigh, *The theory of sound*. Dover, New York 1945.

[25] G.S. Rychkov, *The maximum number of limit cycle of the system* $\dot{x} = y - a_1x^3 - a_2x^5, \dot{y} = -x$ *is two*, Differential'nye Uravneniya **11** (1975), 380–391.

[26] B. Van der Pol, *On relaxation-oscillations*, The London, Edinburgh and Dublin Phil. Mag & J. of Sci **2** (7) (1927), 978–992.

[27] B. Van der Pol and J. Van der Mark, *Frequency demultiplication*, Nature **120** (1927), 363–364.

[28] B. Van der Pol, *The nonlinear theory of electric oscillations*, Proc. IRE **22** (1934), 1051–1085.

[29] Z. Zhang, T. Ding, W. Huang and Z. Dong, *Qualitative theory of differential equations*, Vol. **101** of Trans. of Mathematical Monographs, Amer. Math. Soc., Providence, R.I., 1992.

[1] Departamento de Matemática, Universidade Estadual de Campinas, Rua Sérgio Baruque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083–859, Campinas, SP, Brazil

Email address: candidomr@ime.unicamp.br

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.ist.utl.pt