DETECTING PERIODIC ORBITS IN SOME 3D CHAOTIC QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

TIAGO CARVALHO[1], RODRIGO D. EUZÉBIO[2], JAUME LLIBRE[3]
AND DURVAL JOSÉ TONON[4]

ABSTRACT. Using the averaging theory we study the periodic solutions and their linear stability of the 3–dimensional chaotic quadratic polynomial differential systems without equilibria studied in [3]. All these differential systems depend only on one–parameter.

1. INTRODUCTION

Finding periodic orbits of differential systems takes an important place in the study of the behavior of the trajectories of a given differential system. Indeed, after the equilibrium points, the periodic orbits and their kind of stability provide many information on the dynamics of a differential system, mainly when the system under study models a real problem coming from biology, physics, engineering, etc. But in general the study of the periodic solutions of a differential system is not an easy task because these objects are not local as the equilibrium points.

An effective way to obtain periodic orbits is using the bifurcation theory. Since in many situations the occurrence of periodic orbits is observed by varying a specific parameter of the system, in the sense that it changes the topological phase portrait of the system by assuming some particulars values,the values where the system is not structurally stable. In such bifurcations sometimes it can be observed the appearance of periodic orbits.

In this paper our goal is to apply such approach in order to study the periodic orbits of some families of chaotic differential systems by varying a small specific parameter in a convenient way. However, in our approach we do not make use of the classical bifurcation theory but we apply the *averaging theory* (for more details on it see the Appendix), which is an effective way to study periodic orbits of non–autonomous periodic systems.

More precisely, we study the occurrence of periodic orbits from a list of chaotic quadratic polynomial differential systems in $\mathbb{R}^3$ provided by Jafari, Sprott and Golpayeganiin [3]. In that paper the authors exhibit a list of seventeen 3–dimensional differential systems with quadratic nonlinearities, that they denote by NE_i, $i = 1, \ldots, 17$, all of them depending on one parameter a, and they provide values of a in order to obtain chaotic behavior when the corresponding system has no equilibria. Our goal is to apply the averaging theory in order to study the periodic orbits of such differential systems when the parameter a is sufficiently small. In

2010 *Mathematics Subject Classification.* Primary 34C05.
Key words and phrases. periodic solution, averaging theory, quadratic polynomial differential system.

other words we study the codimension one bifurcation of periodic orbits from those chaotic system differential systems, that is, we provide sufficient conditions in order that some of the systems NE_i, $i = 1, \ldots, 17$ posses periodic motion.

We note that the averaging theory does not provide information for all the differential systems from the list of [3]. The systems of such a list for which this theory provides information are

$$(NE_1) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - yz, \\ \dot{z} &= y^2 - a. \end{aligned}$$

$$(NE_2) \qquad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + z, \\ \dot{z} &= 2y^2 + xz - a. \end{aligned}$$

$$(NE_3) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \frac{1}{10}x^2 + \frac{11}{10}xz + a. \end{aligned}$$

$$(NE_4) \qquad \begin{aligned} \dot{x} &= -\tfrac{1}{10}y + a, \\ \dot{y} &= x + z, \\ \dot{z} &= xz - 3y. \end{aligned}$$

$$(NE_6) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y - xz - yz - a. \end{aligned}$$

$$(NE_7) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + z, \\ \dot{z} &= -\frac{4}{5}x^2 + z^2 + a. \end{aligned}$$

$$(NE_8) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - yz, \\ \dot{z} &= xy + \frac{1}{2}y^2 - a. \end{aligned}$$

$$(NE_9) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - yz, \\ \dot{z} &= -xz + 7x^2 - a. \end{aligned}$$

$$(NE_{11}) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + z, \\ \dot{z} &= z - 2xy - \frac{9}{5}xz - a. \end{aligned}$$

Systems (NE_1), (NE_2) and (NE_3) are modified versions of the *Sprott A system*, *Wei system* and *Wang-Chen system*, respectively (for a detailed analyzes see [3]). In [9] the authors verify the existence of periodic orbits when $a = 1$ coming from infinity for a modified version of (NE_1). However such periodic orbit does not coincide with the periodic orbit that we have found, because our periodic orbit does not come from infinity (see Theorem 1). Other interesting dynamical behaviors of related differential systems with the ones studied here can be found in [2, 4, 5, 6, 8, 9].

In what follows we present our main result.

Theorem 1. *Each one of the systems* (NE_1), (NE_2), (NE_3), (NE_4), (NE_6), (NE_7), (NE_8), (NE_9) *and* (NE_{11}) *presents at least one periodic solution of the form*

(*i*) $x(t) = \sqrt{2a}\cos t + O(\varepsilon)$, $y(t) = -\sqrt{2a}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

(*ii*) $x(t) = \sqrt{a}\cos t + O(\varepsilon)$, $y(t) = \sqrt{a}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

(*iii*) $x(t) = -\sqrt{2a}\cos t + O(\varepsilon)$, $y(t) = \sqrt{2a}\sin t + O(\varepsilon)$, $z(t) = \sqrt{2a}\cos t + O(\varepsilon)$;

(*iv*) $x(t) = -\sqrt{2a}\sin\left(\sqrt{\frac{31}{10}}t\right) + O(\varepsilon)$, $y(t) = 2\sqrt{155a}\cos\left(\sqrt{\frac{31}{10}}t\right) + O(\varepsilon)$, $z(t) = -30\sqrt{2a}\sin\left(\sqrt{\frac{31}{10}}t\right) + O(\varepsilon)$;

(*vi*) $x(t) = \sqrt{2a}\sin t + O(\varepsilon)$, $y(t) = \sqrt{2a}\cos t + O(\varepsilon)$, $z(t) = -\sqrt{2a}\sin t + O(\varepsilon)$;

(*vii*) $x(t) = \sqrt{\frac{5a}{2}}\cos t + O(\varepsilon)$, $y(t) = -\sqrt{\frac{5a}{2}}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

(*viii*) $x(t) = 2\sqrt{a}\cos t + O(\varepsilon)$, $y(t) = -2\sqrt{a}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

(*ix*) $x(t) = \sqrt{\frac{2a}{7}}\cos t + O(\varepsilon)$, $y(t) = -\sqrt{\frac{2a}{7}}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

(*xi*) $x(t) = \sqrt{\frac{5a}{2}}\cos t + O(\varepsilon)$, $y(t) = -\sqrt{\frac{5a}{2}}\sin t + O(\varepsilon)$, $z(t) = O(\varepsilon)$;

where $|\varepsilon| \neq 0$ *is a small parameter and* $a > 0$. *The periodic solutions (i), (iii), (iv), (vi),(vii), (viii) and (ix) are linearly stable; and the periodic solutions (ii) and (xi) are unstable (saddle type).*

Theorem 1 is proved in section 2.

Remark 2. *It is easy to check that the periodic solutions described in Theorem 1 exist when the corresponding differential system have no equilibria, except for system* (iii) *and* (xi) *that when the periodic solution exists there are also equilibria.*

2. Proof of Theorem 1

In this section we prove our main result. The proofs are separated by the statements of Theorem 1. We start proving statement $(viii)$ of Theorem 1 in subsection 2.1 and then, in subsection 2.2, we only provide the periodic solution of items (i) and (ix), because their proofs are analogous to the proof of statement $(viii)$. The proof statement (ii) in subsection 2.3, and we do not provide the complete proofs of statements (iii), (vi) and (vii) in subsection 2.4 because are similar to the proof of statement (ii).

Moreover, in every proof we shall consider $a = \varepsilon^2 b$, for indicating that a goes to zero faster than ε. Note that the averaging method is performed with the small parameter ε. We also consider a re-escale of variables using again the small parameter ε, which allows to reach the hypotheses of the averaging method, although this re-escale of a and the variables is simply a technical trick in order to apply the averaging theory. The results are obtained for the original systems (NE_k) for $k = 1, 2, 3, 4, 6, 7, 8, 9, 11$. Similar re–scaling were used in [1].

Now we shall prove the results.

2.1. **Proof of statement** $(viii)$ **of Theorem 1.** Consider $a = \varepsilon^2 b$ in system (NE_8). Since the main tool for proving our results is the averaging theory, we need to transform the differential system (NE_8) into the differential system of the normal form (16) for applying the averaging theory, see the Appendix. Thus, first we re-scale the variables as follows $(x, y, z) = (\varepsilon\overline{x}, \varepsilon\overline{y}, \varepsilon\overline{z})$, then system (NE_8) becomes

$$\begin{aligned} \dot{\overline{x}} &= \overline{y}, \\ \dot{\overline{y}} &= -\overline{x} - \varepsilon\overline{yz}, \\ \dot{\overline{z}} &= \varepsilon\left(-b + \tfrac{1}{2}\overline{x}^2 + \overline{xy}\right). \end{aligned} \tag{1}$$

Now we write this differential system in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, and we get the differential system

$$\begin{aligned} \dot{r} &= \quad -\varepsilon r z \sin^2\theta, \\ \dot{\theta} &= \quad -1 - \varepsilon z \cos\theta \sin\theta, \\ \dot{z} &= \quad -\varepsilon \tfrac{1}{2}(2b - r^2\cos^2\theta - 2r^2\cos\theta\sin\theta). \end{aligned} \tag{2}$$

Now we take as new independent variable in the differential system (2) the variable θ and this system can be written as

$$\begin{aligned} r' &= \quad \varepsilon r z \sin^2\theta + O(\varepsilon^2), \\ z' &= \quad \varepsilon \tfrac{1}{2}(2b - r^2\cos^2\theta - 2r^2\cos\theta\sin\theta) + O(\varepsilon^2). \end{aligned} \tag{3}$$

Here the prime denotes derivative with respect to the variable θ.

Note that the differential system (3) is written into the normal form (16) for applying the averaging theory, see for more details the Appendix. Moreover, it is nonautonomous and 2π-periodic. Now, using the notation of the Appendix we have

$$x = (r, z), \quad t = \theta, \quad F_1(t, x) = F_1(\theta, r, z),$$

where

$$F_1(r, \theta, z) = \begin{pmatrix} F_{11}(r, \theta, z) \\ F_{12}(r, \theta, z) \end{pmatrix} = \begin{pmatrix} r z \sin^2\theta \\ \frac{1}{2}(2b - r^2\cos^2\theta - 2r^2\cos\theta\sin\theta) \end{pmatrix}.$$

Now we consider the averaging function (18) of the Appendix

$$f(r, z) = \begin{pmatrix} f_1(r, z) \\ f_2(r, z) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F_{11}(r, \theta, z) d\theta \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F_{12}(r, \theta, z) d\theta \end{pmatrix} = \begin{pmatrix} \dfrac{rz}{2} \\ \dfrac{4b - r^2}{4} \end{pmatrix}.$$

The averaged function $f(r, z)$ has a unique zero with $r > 0$, namely $\widetilde{z} = (2\sqrt{b}, 0)$ if $b > 0$. The Jacobian (3) of the function f at $\widetilde{z}$ is b.

On the other hand, the eigenvalues associated to the zero $\widetilde{z}$ are $-\sqrt{-b}$ and $\sqrt{-b}$. Therefore according to statement (b) of Theorem 3 the periodic solution of system (3) is linearly stable.

Now we go back through the changes of variables in order to estimate in the initial variables (x, y, z) how is the periodic orbit that we have obtained if $b > 0$.

According with statement (a) of Theorem 3 the periodic solution of system (3) associated to the zero $\widetilde{z}$ is of the form

$$r(\theta, \varepsilon) = 2\sqrt{b} + O(\varepsilon), \quad z(\theta, \varepsilon) = O(\varepsilon).$$

and in the differential system (2) becomes

$$r(t,\varepsilon) = 2\sqrt{b} + O(\varepsilon), \quad \theta(t,\varepsilon) = -t + O(\varepsilon), \quad z(\theta,\varepsilon) = O(\varepsilon).$$

This periodic solution in the differential system (1) writes

$$\overline{x}(t,\varepsilon) = 2\sqrt{b}\cos t + O(\varepsilon), \quad \overline{y}(t,\varepsilon) = -2\sqrt{b}\sin t + O(\varepsilon), \quad \overline{z}(t,\varepsilon) = O(\varepsilon).$$

Finally for the differential system (NE_8) this last periodic solution writes

$$x(t,\varepsilon) = \varepsilon 2\sqrt{b}\cos t + O(\varepsilon^2), \quad y(t,\varepsilon) = -\varepsilon 2\sqrt{b}\sin t + O(\varepsilon^2), \quad z(t,\varepsilon) = O(\varepsilon^2).$$

or equivalently

$$x(t,\varepsilon) = 2\sqrt{a}\cos t + O(\varepsilon), \quad y(t,\varepsilon) = -2\sqrt{a}\sin t + O(\varepsilon), \quad z(t,\varepsilon) = O(\varepsilon).$$

Therefore we have proved item $(viii)$ of Theorem 1.

2.2. **Proof of statements** (i) **and** (ix) **of Theorem 1.** Doing exactly the same computations than in the previous subsection for the systems (NE_1) and (NE_9) we get the periodic solutions of statements (i) and (ix) of Theorem 1.

2.3. **Proof of statement** (ii) **of Theorem 1.** Again we take $a = \varepsilon^2 b$. The proof of this statement is slightly different from the previous proofs because we must perform a change of variables in order to write the linear part of system (NE_2) with $\varepsilon = 0$ in its real Jordan normal form, and another change in order to be in the assumptions for applying the averaging theory.

First, as in the proof of statement $(viii)$, we re-scale the variables as follows $(x,y,z) = (\varepsilon\overline{x}, \varepsilon\overline{y}, \varepsilon\overline{z})$, then system ($NE_2$) becomes

$$\begin{aligned} \dot{\overline{x}} &= -\overline{y}, \\ \dot{\overline{y}} &= \overline{x} + \overline{z}, \\ \dot{\overline{z}} &= \varepsilon\left(2\overline{y}^2 + \overline{xz} - b\right). \end{aligned} \tag{4}$$

The linear part of this system at the origin when $\varepsilon = 0$ is

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

we write the linear part (5) in its real Jordan normal form, and we get the differential system

$$\begin{aligned} \dot{u} &= -v + \varepsilon\left(-b + 2v^2 + uw - w^2\right), \\ \dot{v} &= u, \\ \dot{w} &= \varepsilon\left(-b + 2v^2 + uw - w^2\right). \end{aligned} \tag{6}$$

We write this differential system in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, and we get the differential system

$$
\begin{aligned}
\dot{r} &= \varepsilon\cos\theta(-b - w^2 + rw\cos\theta + 2r^2\sin^2\theta),\\
\dot{\theta} &= 1 + \varepsilon\frac{\sin\theta}{r}(b - r^2\sin^2\theta + w^2 - rw\cos\theta),\\
\dot{w} &= \varepsilon(-b - w^2 + rw\cos\theta + 2r^2\sin^2\theta).
\end{aligned}
\tag{7}
$$

Now taking θ as the new independent variable in the differential system it becomes

$$
\begin{aligned}
r' &= \varepsilon\cos\theta(-b - w^2 + rw\cos\theta + 2r^2\sin^2\theta) + O(\varepsilon^2)\\
&= \varepsilon F_{11}(\theta, r, w) + O(\varepsilon^2),\\
w' &= \varepsilon(-b - w^2 + rw\cos\theta + 2r^2\sin^2\theta) + O(\varepsilon^2)\\
&= \varepsilon F_{12}(\theta, r, w) + O(\varepsilon^2).
\end{aligned}
\tag{8}
$$

Note that system (8) is written into the standard form (16) for applying the averaging method, it is a non–autonomous 2π–periodic differential system. So, using the same notation than in the Appendix we get the averaged function (18) of the Appendix is

$$
f(r,w) = \begin{pmatrix} f_1(r,w) \\ f_2(r,w) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^{2\pi} F_{11}(r,\theta,w)d\theta \\ \displaystyle\int_0^{2\pi} F_{12}(r,\theta,w)d\theta \end{pmatrix},
$$

where

$$
f_1(r,w) = \frac{rw}{2} \quad \text{and} \quad f_2(r,w) = -b + r^2 - w^2.
$$

Consequently the zero of the averaging function $f(r,w)$ satisfying $r > 0$ is $(\sqrt{b}, 0)$ with $b > 0$. Since the determinant of Jacobian matrix of the function f at this zero is $-b \neq 0$, by Theorem 3 this zero provides, for $\varepsilon \neq 0$ sufficiently small, the following periodic solution of system (8)

$$
r(\theta,\varepsilon) = \sqrt{b} + O(\varepsilon), \quad w(\theta,\varepsilon) = O(\varepsilon).
\tag{9}
$$

The eigenvalues of the Jacobian matrix of the function f at the zero $(\sqrt{b}, 0)$ are $\pm\sqrt{b}$ if $b > 0$. Hence the periodic orbit is unstable if $b > 0$.

The periodic solution (9) in the differential system (7) becomes

$$
r(t,\varepsilon) = \sqrt{b} + O(\varepsilon), \quad \theta(t,\varepsilon) = t + O(\varepsilon), \quad w(t,\varepsilon) = O(\varepsilon).
$$

Consequently the periodic solution in coordinates (u, v, w) for system (6) writes

$$
u(t,\varepsilon) = \sqrt{b}\cos t + O(\varepsilon), \quad v(t,\varepsilon) = \sqrt{b}\sin t + O(\varepsilon), \quad w(t,\varepsilon) = O(\varepsilon).
$$

In the coordinates $(\overline{x}, \overline{y}, \overline{z})$ of system 4 the periodic solution becomes

$$
\overline{x}(t,\varepsilon) = \sqrt{b}\cos t + O(\varepsilon), \quad \overline{y}(t,\varepsilon) = \sqrt{b}\sin t + O(\varepsilon), \quad \overline{z}(t,\varepsilon) = O(\varepsilon).
$$

Undoing the re–scaling we get the periodic solution

$$
x(t,\varepsilon) = \varepsilon\sqrt{b}\cos t + O(\varepsilon^2), \quad y(t,\varepsilon) = \varepsilon\sqrt{b}\sin t + O(\varepsilon^2), \quad z(t,\varepsilon) = O(\varepsilon^2),
$$

for the system (NE_2). Finally substituting the parameter b for the parameter a we obtain the periodic solutions writes

$$x(t) = \sqrt{a}\cos t + O(\varepsilon), \quad y(t) = \sqrt{a}\sin t + O(\varepsilon), \quad z(t) = O(\varepsilon).$$

Hence statement (ii) is proved.

2.4. **Proof of statements** (iii)**,** (iv)**,** (vi) **and** (vii) **of Theorem 1.** Repeating the proof of statement (ii) for the systems (NE_3), (NE_6) and (NE_7) we get for them the periodic solutions which appear in the statements (iii), (iv), (vi) and (vii) of Theorem 1.

2.5. **Proof of statement** (xi) **of Theorem 1.** We consider $a = \varepsilon^2 b$. The proof of this statement again is slightly different from the previous proofs because additionally to do a change of variables for writing the linear part of system (NE_{11}) with $\varepsilon = 0$ in its real Jordan normal form, we need an additional special change of variables in order to get the normal form (16) for applying the averaging theory.

As in the previous proofs we re-scale the variables as follows $(x, y, z) = (\varepsilon\overline{x}, \varepsilon\overline{y}, \varepsilon\overline{z})$, then system (NE_{11}) becomes

$$\begin{aligned}\dot{\overline{x}} &= \overline{y},\\ \dot{\overline{y}} &= -\overline{x} + \overline{z},\\ \dot{\overline{z}} &= \overline{z} + \varepsilon\left(-2\overline{xy} + \tfrac{9}{5}\overline{xz} - b\right).\end{aligned} \tag{10}$$

The linear part of this system at the origin when $\varepsilon = 0$ is

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

we write the linear part (11) in its real Jordan normal form, and we get the differential system

$$\begin{aligned}\dot{u} &= -v + \varepsilon\frac{1}{10}\left(5b - 10uv - 14uw + 5vw + 7w^2\right),\\ \dot{v} &= u + \varepsilon\frac{1}{10}\left(-5b + 10uv + 14uw - 5vw - 7w^2\right),\\ \dot{w} &= w + \varepsilon\frac{1}{5}\left(5b - 10uv - 14uw + 5vw + 7w^2\right).\end{aligned} \tag{12}$$

We write this differential system in cylindrical coordinates $u = r\cos\theta$, $v = r\sin\theta$, and $w = w$, and we get the differential system

$$
\begin{aligned}
\dot r =&\ \varepsilon\frac{1}{10}(\sin\theta-\cos\theta)\big(-5b+10r^2\sin\theta\cos\theta-5rw\sin\theta+\\
&\qquad 14rw\cos\theta-7w^2\big),\\
\dot\theta =&\ 1+\varepsilon\frac{1}{10r}(\sin\theta+\cos\theta)\big(-5b+10r^2\sin\theta\cos\theta-5rw\sin\theta+\\
&\qquad 14rw\cos\theta-7w^2\big),\\
\dot w =&\ w+\varepsilon\frac{1}{5}\big(5b-10r^2\sin\theta\cos\theta+5rw\sin\theta-14rw\cos\theta+7w^2\big).
\end{aligned}
\tag{13}
$$

Now taking θ as the new independent variable in the differential system it becomes

$$
\begin{aligned}
r' =&\ \varepsilon\frac{1}{10}(\sin\theta-\cos\theta)\big(-5b+10r^2\sin\theta\cos\theta-5rw\sin\theta+\\
&\qquad 14rw\cos\theta-7w^2\big)+O(\varepsilon^2),\\
w' =&\ w+\varepsilon\frac{1}{10r}\big(-5b+10r^2\sin\theta\cos\theta-5rw\sin\theta+14rw\cos\theta-7w^2\big)\\
&\qquad \big(2r+(\cos\theta+\sin\theta)w\big)+O(\varepsilon^2).
\end{aligned}
\tag{14}
$$

Finally doing the change of variables $R = r$ and $W = e^\theta w$ system (14) writes

$$
\begin{aligned}
R' =&\ \varepsilon\frac{1}{10}(\sin\theta-\cos\theta)\big(-5b+10R^2\sin\theta\cos\theta\\
&-5e^\theta RW\sin\theta+14e^\theta RW\cos\theta-7e^{2\theta}W^2\big)+O(\varepsilon^2)\\
=&\ \varepsilon F_{11}(\theta,r,w)+O(\varepsilon^2),\\
W' =&\ \varepsilon\frac{1}{10R}e^{-\theta}\big(2R+e^\theta W(\cos\theta+\sin\theta)\big(5b+7e^{2\theta}W^2\\
&+5e^\theta RW\sin\theta-2R\cos\theta\big)\big(7e^\theta W+5R\sin\theta\big)+O(\varepsilon^2)\\
=&\ \varepsilon F_{12}(\theta,r,w)+O(\varepsilon^2),
\end{aligned}
\tag{15}
$$

Note that system (15) is written into the standard form (16) for applying the averaging method, it is a non–autonomous 2π–periodic differential system. So, using the same notation than in the Appendix we get that the averaged function (18) of the Appendix is $f(R,W) = (f_1(R,W), f_2(R,W))$ where

$$
\begin{aligned}
f_1(R,W) &= \frac{\left(e^{2\pi}-1\right)W\left(21\left(1+e^{2\pi}\right)W-71R\right)}{100\pi},\\
f_2(R,W) &= \frac{e^{-2\pi}\left(e^{2\pi}-1\right)}{100\pi R}\left(50bR-20R^3+47e^{2\pi}RW^2+7\left(e^{2\pi}+e^{4\pi}\right)W^3\right).
\end{aligned}
$$

The averaging function $f(R,W)$ has two zeros, namely

$$
\left(\sqrt{\frac{5b}{2}},0\right)\ \text{and}\ \left(\frac{105\sqrt{b}\cosh\pi}{\sqrt{2205\cosh(2\pi)-253943/6}},\frac{355\,e^{-\pi}\sqrt{b}}{\sqrt{8820\cosh(2\pi)-507886/3}}\right),
$$

with $b > 0$. The second zero provides a periodic solution for the differential system (15), but when we go back to system (14) such solution is not more periodic due to the change of variables where appears e^θ. But the first zero provides a periodic solution also for system (14) due to the fact that in this zero $W = 0$ and cancels the function e^θ which appears in the change of variables for passing from system (15) to system (14). Of course, the absolute value determinant of the Jacobian matrix of the function $f(R,W)$ at both zeros is

$$\frac{71b\sinh^2\pi}{25\pi^2}\neq 0.$$

Moreover, the eigenvalues of the Jacobian matrix at the first zero are

$$\pm\frac{e^{-\pi}\sqrt{71b}\,(e^{2\pi}-1)}{10\pi}.$$

Then the periodic orbit associated to this zero is unstable of type saddle.

Now we go back through the changes of variables in order to estimate the coordinates of the periodic solution in the initial variables (x,y,z). According to statement (a) of Theorem 3 the periodic solution of system (15) associated to the first zero is of the form

$$R(\theta,\varepsilon)=\sqrt{\frac{5b}{2}}+O(\varepsilon),\quad W(\theta,\varepsilon)=O(\varepsilon).$$

This periodic solution in system (14) writes

$$r(\theta,\varepsilon)=\sqrt{\frac{5b}{2}}+O(\varepsilon),\quad w(\theta,\varepsilon)=O(\varepsilon),$$

and in the differential system (13) becomes

$$r(t,\varepsilon)=\sqrt{\frac{5b}{2}}+O(\varepsilon),\quad \theta(t,\varepsilon)=t+O(\varepsilon),\quad w(t,\varepsilon)=O(\varepsilon).$$

This periodic solution in the differential system (12) writes

$$u(t,\varepsilon)=\sqrt{\frac{5b}{2}}\cos t+O(\varepsilon),\quad v(t,\varepsilon)=\sqrt{\frac{5b}{2}}\sin t+O(\varepsilon),\quad w(t,\varepsilon)=O(\varepsilon),$$

and in the differential system (10) becomes

$$\overline{x}(t,\varepsilon)=\sqrt{\frac{5b}{2}}\cos t+O(\varepsilon),\quad \overline{y}(t,\varepsilon)=-\sqrt{\frac{5b}{2}}\sin t+O(\varepsilon),\quad \overline{z}(t,\varepsilon)=O(\varepsilon).$$

Finally undoing the re–scaling and changing the parameter b by the parameter a we get the periodic solution for the differential system (NE_{11}) we get the periodic solution

$$x(t,\varepsilon)=\sqrt{\frac{5a}{2}}\cos t+O(\varepsilon),\quad y(t,\varepsilon)=\sqrt{\frac{5a}{2}}\sin t+O(\varepsilon),\quad z(t,\varepsilon)=O(\varepsilon).$$

with $a>0$.

3. Conclusion

By using averaging theory we have shown the existence of periodic solutions for some chaotic systems from the list presented by Jafari Sprott and Golpayegani in [3], namely, for systems (NE_1), (NE_2), (NE_3), (NE_4), (NE_6), (NE_7), (NE_8), (NE_9) and (NE_{11}). Moreover, we have characterized the linear stability of all the periodic solutions that we found. The approach involves the replacement of the parameter a and the variables x, y and z for new ones depending on a small parameter ε, and then the appliance of the first order averaging method.

Appendix: The averaging theory of first order

Consider the differential system

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon), \quad x(0) = x_0 \tag{16}$$

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. We also assume that the functions $F_1(t,x)$ and $F_2(t,x,\varepsilon)$ are $T-$periodic in t. We define in D the averaged differential system

$$\dot{y} = \varepsilon f(y), \quad y(0) = x_0, \tag{17}$$

where

$$f(y) = \frac{1}{T}\int_0^T F_1(t,y)dt. \tag{18}$$

The next result shows that under the convenient hypotheses the equilibria of the averaged system will provide $T-$periodic solutions of system (16).

Theorem 3. *Consider the two initial value problems* (16) *and* (17)*. Suppose that*

(i) *the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 and $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$;*

(ii) *the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged system* (17) *satisfying*

$$\det\left(\frac{\partial f}{\partial y}\right)\bigg|_{y=p} \neq 0,$$

then there is a $T-$periodic solution $x(t,\varepsilon)$ of system (16) *such that $x(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *The kind of linear stability or instability of the periodic solution $x(t,\varepsilon)$ coincides with the kind of stability or instability of the equilibrium point p of the averaged system* (17)*. The equilibrium point p has the kind of stability behavior of the Poincaré map associated to the periodic solution $x(t,\varepsilon)$.*

For a proof of Theorem 3, see sections 6.3 and 11.8 of Verhulst [10].

Acknowledgements

The first author is supported by grant number 2014/02134-7, São Paulo Research Foundation (FAPESP). The second author is supported by grant number 2013/25828-1, São Paulo Research Foundation (FAPESP).

The third author is partially supported by a MINECO/FEDER grant MTM 2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014SGR568, an ICREA Academia and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, and the grant UNAB13-4E-1604.

The first and the third authors are partially supported by a CAPES/Brazil grant number 88881.030454/ 2013-01 from the program CSF-PVE.

The fourth author is supported by FAPEG-Brazil Project 2012/10 26 7000 803.

The first and the fourth authors are partially supported by the CNPq/Brazil grants 478230/2013-3 and 443302/2014-6. This work is partially realized at UFG as a part of project numbers 35796, 35798 and 040393.

References

[1] R.D. Euzébio and J. Llibre, *Periodic Solutions of El Niño model through the Vallis differential system*, DCDS-A **34** (2014), 3455–3469.

[2] S. Jafari and J.C. Sprott, *Simple chaotic flows with a line equilibrium*, Chaos Solit. Fract. **57** (2013), 79–84.

[3] S. Jafari, J.C. Sprott, S.M.R.H. Golpayegani, *Elementary quadratic chaotic flows with no equilibria*, Physics Letters A **377** (2013), 699–702.

[4] F. Legoll, M. Luskin and R. Moeckel, *Non-ergodicity of Nosé–Hoover dynamics*, Nonlinearity **22** (2009), 1673–1694.

[5] A. Mahdi and C. Valls, *Integrability of the Nosé–Hoover equation*, J. Geo. Phy. **61** (2011), 1348–1352.

[6] V.T. Pham, C. Volos, S. Jafari, Z. Wei and X. Wang, *Constructing a Novel No–Equilibrium Chaotic System*, Int. J. Bifuc. Chaos **24** (2014), 1450073, pp 6.

[7] J.A. Sanders, F. Verhulst and J. Murdock, *Averaging methods in nonlinear dynamical systems*, Second edition, Applied Mathematical Sciences **59**, Springer, New York, 2007.

[8] J.C. Sprott, *Some simple chaotic flows*, Phys. Rev. A **3** (1994), 647–650.

[9] P. Swinnerton-Dyer and T. Wagenknecht, *Some third-order ordinary differential equations*, London Mathematical Society **40** (2008), 725–748.

[10] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext Springer Verlag, 1996.

[1] Departamento de Matemática, Faculdade de Ciências, UNESP, Av. Eng. Luiz Edmundo Carrijo Coube 14-01, CEP 17033-360, Bauru, SP, Brazil.

E-mail address: `tcarvalho@fc.unesp.br`

[2] Departamento de Matemática, IMECC–UNICAMP, CEP 13083–970, Campinas, SP, Brazil

E-mail address: `euzebio@ime.unicamp.br`

[3] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.

E-mail address: `jllibre@mat.uab.cat`

[4] Universidade Federal de Goiás, IME, CEP 74001-970, Caixa Postal 131, Goiânia, Goiás, Brazil.

E-mail address: `djtonon@ufg.br`