Nonlinear Dyn (2017) 88:547–553
DOI 10.1007/s11071-016-3259-2

ORIGINAL PAPER

Transcritical and zero-Hopf bifurcations in the Genesio system

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Received: 11 July 2016 / Accepted: 2 December 2016 / Published online: 27 December 2016

Abstract In this paper we study the existence of transcritical and zero-Hopf bifurcations of the third-order ordinary differential equation $\dddot{x}+a\ddot{x}+b\dot{x}+cx-x^2=0$, called the Genesio equation, which has a unique quadratic nonlinear term and three real parameters. More precisely, writing this differential equation as a first-order differential system in $\mathbb{R}^3$ we prove: first that the system exhibits a transcritical bifurcation at the equilibrium point located at the origin of coordinates when $c=0$ and the parameters (a,b) are in the set $\{(a,b)\in\mathbb{R}^2 : b\neq 0\}\backslash\{(0,b)\in\mathbb{R}^2 : b>0\}$, and second that the system has a zero-Hopf bifurcation also at the equilibrium point located at the origin when $a=c=0$ and $b>0$.

Keywords Genesio system · Transcritical bifurcation · Zero-Hopf Bifurcation · Averaging theory

Mathematics Subject Classification 34C23 · 34C25 · 37G10

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1 Introduction

In [4] Genesio and Tesi, inspired by the problem of determining conditions under which a nonlinear dynamical system presents chaotic behavior, introduced the following third-order ordinary differential Eq.

$$\dddot{x}+a\ddot{x}+b\dot{x}+cx-x^2=0, \tag{1}$$

where a, b and c are parameters and the dot indicates derivative with respect to the time t. If we define $y=\dot{x}$ and $z=\dot{y}$ the differential Eq. (1) becomes the first-order differential system

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=z,\\ \dot{z}&=-cx-by-az+x^2,\end{aligned} \tag{2}$$

which is commonly known as the *Genesio system*. Based on the harmonic balance principle the authors of [4] presented two practical methods for predicting the existence and the location of chaotic motions. For instance, system (2) exhibits chaotic dynamical behaviors when $a=1.2$, $b=2.92$ and $c=6$.

We can find in the literature several articles concerning system (2). For instance, issues on synchronization of Genesio chaotic system have been studied in the articles [3,9,10,15]. Already in [16] the authors studied the Hopf bifurcation and the existence of Silnikov homoclinic orbit for this system. Stability analysis and