LIMIT CYCLES OF DISCONTINUOUS PIECEWISE POLYNOMIAL VECTOR FIELDS

TIAGO DE CARVALHO[1], JAUME LLIBRE[2] AND DURVAL JOSÉ TONON[3]

Abstract. When the first average function is non-zero we provide an upper bound for the maximum number of limit cycles bifurcating from the periodic solutions of the center $\dot{x} = -y((x^2+y^2)/2)^m$ and $\dot{y} = x((x^2+y^2)/2)^m$ with $m \geq 1$, when we perturb it inside a class of discontinuous piecewise polynomial vector fields of degree n with k pieces. The positive integers m, n and k are arbitrary. The main tool used for proving our results is the averaging theory for discontinuous piecewise vector fields.

1. Introduction and statement of the main result

One of the main problems inside the qualitative theory of real planar differential systems is the determination of their limit cycles. The notion of a *limit cycle* of a planar differential system was defined by Poincaré [27], as a periodic orbit isolated in the set of all periodic orbits of the differential system. Van der Pol [28], Liénard [20] and Andronov [1] at the end of 1920s proved that a periodic orbit of a self–sustained oscillation occurring in a vacuum tube circuit was a limit cycle in the sense defined by Poincaré. After these results on the existence, non-existence and other properties of the limit cycles, these were studied with interest by mathematicians and physicists, and more recently also by many scientists of different areas (see for instance the books [10, 32]).

In the last part of the XIX century Poincaré [27] defined the notion of a *center* of a real planar differential system, i.e. of an isolated equilibrium point having a neighbourhood such that all the orbits of this neighbourhood are periodic with the unique exception of the equilibrium point. Later on one way to produce limit cycles is by perturbing the periodic orbits of a center [29].

Iliev [19] in 1999 considered the polynomial vector fields

$$\mathcal{X}(x,y) = \big(-y + \varepsilon P(x,y,\varepsilon), x + \varepsilon Q(x,y,\varepsilon)\big),$$

of degree $n > 1$ (i.e. the maximum of the degrees of polynomials P and Q is n), which depend analytically on the small parameter ε, and he studied how many limit cycles can bifurcate from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ when $\varepsilon > 0$ is sufficiently small.

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Buică, Giné and Llibre [8] in 2010 studied the same problem of Iliev but for the polynomial vector fields

$$\mathcal{X}(x,y)=\left(-y\left(\frac{x^2+y^2}{2}\right)^m+\varepsilon P(x,y,\varepsilon),x\left(\frac{x^2+y^2}{2}\right)^m+\varepsilon Q(x,y,\varepsilon)\right),$$

of degree the maximum of $2m+1$ and n being the maximum of the degrees of the polynomials P and Q, where again ε is a small parameter, and $m \geq 1$ is an integer. Of course, now the limit cycles bifurcate from the periodic solutions of the nonlinear center $\dot{x}=-y((x^2+y^2)/2)^m$, $\dot{y}=x((x^2+y^2)/2)^m$.

Andronov, Vitt and Khaikin [2] started the study of the continuous and discontinuous piecewise differential systems. These systems play an important role inside the nonlinear dynamical systems. They appeared in a natural way in nonlinear engineering models, and later on in electronic engineering, nonlinear control systems, biology, ... see for instance the books of di Bernardo, Budd, Champneys and Kowalczyk [5], Simpson [31], and the survey of Makarenkov and Lamb [26], and the hundreds of references quoted in these last three works.

There are many studies of the limit cycles of continuous and discontinuous piecewise differential systems in $\mathbb{R}^2$ with two pieces separated by a straight line. In general these differential systems are linear, see for instance [3, 6, 9, 11, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 30]. But there are very few works of continuous and discontinuous piecewise differential systems with an arbitrary number k of pieces.

The objective of this paper is to study the number of limit cycles which can bifurcate from the center $\dot{x}=-y((x^2+y^2)/2)^m$, $\dot{y}=x((x^2+y^2)/2)^m$, when it is perturbed inside a class of discontinuous piecewise polynomial differential systems of degree n with k pieces.

More precisely, we consider the polynomial planar vector field

$$\mathcal{X}=\mathcal{X}(x,y)=\left(-y\left(\frac{x^2+y^2}{2}\right)^m,x\left(\frac{x^2+y^2}{2}\right)^m\right),$$

with either $m=0$ (linear center) or m a positive integer (nonlinear center), and we perturb $\mathcal{X}$ with a discontinuous piecewise polynomial vector field as follows

$$\mathcal{X}_\varepsilon=\mathcal{X}_\varepsilon(x,y)=\mathcal{X}(x,y)+\varepsilon\sum_{i=1}^{k}\chi_{S_i}(x,y)\left(P_i(x,y),Q_i(x,y)\right),$$

where P_i and Q_i are polynomials of degree at most n, the characteristic function χ_K of a set $K\subset\mathbb{R}^2$ is defined by

$$\chi_K(x,y)=\begin{cases}1 & \text{if } (x,y)\in K,\\ 0 & \text{if } (x,y)\notin K,\end{cases}$$

and the sets $S_1,\ldots,S_k$ satisfying $\cup_{i=1}^{k}\overline{S_i}=\mathbb{R}^2$ and $S_i\cap S_j=\emptyset$ for $i\neq j$ are defined as follows. For a given positive integer k consider k angles $0\leq\theta_1<\ldots<\theta_k<2\pi$. Then the discontinuity set Σ for the discontinuous piecewise polynomial differential vector field $\mathcal{X}_\varepsilon$ is $\Sigma=\cup_{i=1}^{k}L_i$, where L_i is the ray starting at the origin and passing through the point $(\cos\theta_i, \sin\theta_i)$ for $i = 1, \ldots, k$, S_i is the interior of the sector with boundaries the rays L_{i-1} and L_i going from L_{i-1} to L_i in counterclockwise sense, and S_1 is the interior of the sector with boundaries the rays L_k and L_1 going from L_k to L_1 in counterclockwise sense. See Figure 1.

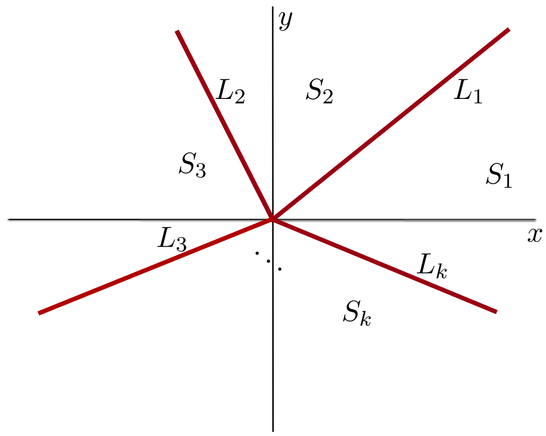

FIGURE 1. The sectors S_i and the rays L_i, for $i = 1, \ldots, k$.

The main result of this paper is the following. For a definition of average function of first order see Section 2.

Theorem 1. *Assume that the average function of first order associated to the discontinuous piecewise polynomial differential system $\mathcal{X}_\varepsilon$ is non-zero. Then, for $\varepsilon > 0$ sufficiently small the maximum number of limit cycles of $\mathcal{X}_\varepsilon$ is n. Moreover, this upper bound is reached.*

Theorem 1 is proved in Section 2. Note that the maximum number n of limit cycles stated in Theorem 1 does not depend on the numbers m and k, i.e., of pieces of the discontinuous piecewise polynomial differential system $\mathcal{X}_\varepsilon$.

In Section 3 we provide some numerical examples of discontinuous polynomial vector fields $\mathcal{X}_\varepsilon$ presenting the maximum number of limit cycles stated in Theorem 1.

We provide in the appendix a summary about the averaging theory for computing periodic solutions of discontinuous piecewise vector fields that we shall use for proving Theorem 1.

2. Proof of Theorem 1

We write the polynomials P_i and Q_i which appear in the definition of the vector field $\mathcal{X}_\varepsilon$ as

$$P_i(x,y) = \sum_{s=0}^{n}\sum_{j=s}^{n} a^i_{s,j-s} x^s y^{j-s} \quad \text{and} \quad Q_i(x,y) = \sum_{s=0}^{n}\sum_{j=s}^{n} b^i_{s,j-s} x^s y^{j-s},$$

with for $i = 1, \ldots, k$. Doing the change of variables $(x, y) \mapsto (\theta, \rho)$, where (θ, ρ) are the polar coordinates defined by $x = \rho\cos\theta$ and $y = \rho\sin\theta$, with $\rho > 0$. Then the differential equation associated to the vector field $\mathcal{X}_\varepsilon$ in polar coordinates is

$$
\begin{aligned}
\dot{\rho} &= \varepsilon \sum_{i=1}^{k} \big(\cos\theta(\chi_{S_i} \cdot P_i)(\rho\cos\theta, \rho\sin\theta) + \sin\theta(\chi_{S_i} \cdot Q_i)(\rho\cos\theta, \rho\sin\theta)\big),\\
\dot{\theta} &= \frac{\rho^{2m}}{2^m} + \frac{\varepsilon}{\rho} \sum_{i=1}^{k} \big(\cos\theta(\chi_{S_i} \cdot Q_i)(\rho\cos\theta, \rho\sin\theta)) - \sin\theta(\chi_{S_i} \cdot P_i)(\rho\cos\theta, \rho\sin\theta)\big).
\end{aligned}
$$

Taking θ as the new independent variable the previous differential system becomes the differential equation

$$
\frac{d\rho}{d\theta} = \varepsilon F(\theta, \rho) + \mathcal{O}(\varepsilon^2), \tag{1}
$$

where

$$
F(\theta, \rho) = \frac{2^m}{\rho^{2m}} \sum_{i=1}^{k} \big(\cos\theta(\chi_{S_i} \cdot P_i)(\rho\cos\theta, \rho\sin\theta) + \sin\theta(\chi_{S_i} \cdot Q_i)(\rho\cos\theta, \rho\sin\theta)\big).
$$

Therefore, from the appendix, the average function associated to the differential equation (1) is

$$
\begin{aligned}
f_n(\rho) = & \int_0^{2\pi} F(\theta, \rho)\, d\theta\\
= & \frac{2^m}{\rho^{2m}} \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \Big(\cos\theta\, P_i(\rho\cos\theta, \rho\sin\theta) + \sin\theta\, Q_i(\rho\cos\theta, \rho\sin\theta)\Big)\, d\theta\\
= & \frac{2^m}{\rho^{2m}} \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \Bigg(\cos\theta \sum_{s=0} \sum_{j=s} a^i_{s,j-s} \rho^j \cos^s\theta \sin^{j-s}\theta\\
& + \sin\theta \sum_{s=0} \sum_{j=s} b^i_{s,j-s} \rho^j \cos^s\theta \sin^{j-s}\theta \Bigg)\, d\theta.
\end{aligned}
$$

From the summary on the averaging theory for the discontinuous piecewise differential equations of the form (1) given in the appendix, we know that for $\varepsilon > 0$ sufficiently small each simple zero of the average function $f_n(\rho)$ provides a limit cycle of the differential equation (1). In order to study the simple zeros of the function $f_n(\rho)$ we shall apply the *Descartes's Theorem*. We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [4]).

Descartes Theorem. *Consider the real polynomial* $p(\rho) = a_{i_1}\rho^{i_1} + a_{i_2}\rho^{i_2} + \cdots + a_{i_r}\rho^{i_r}$ *with* $0 \le i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{0, 1, 2, \cdots, r\}$. *When* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(\rho)$ *has at most* m *positive real roots. Moreover,* *it is always possible to choose the coefficients of $p(\rho)$ in such a way that $p(\rho)$ has exactly r positive real roots.*

In the next lemma we prove that the averaged function $f_n(\rho)$ is generated by a linear combination of a set $\{g_0(\rho), g_1(\rho), \ldots, g_n(\rho)\}$ of linearly independent functions.

Lemma 2. *The average function $f_n(\rho)$ is a linear combination of the linearly independent set of functions $\mathcal{F}_n = \{\rho^{-2m}, \rho^{-2m+1}, \ldots, \rho^{-2m+n}\}$. More precisely, $f_n(\rho) = \sum_{r=0}^{n} 2^m A_r \rho^{-2m+r}$ where*

$$A_r = \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \sum_{s=0}^{r} \left(a^i_{s,r-s} \cos^{s+1}\theta \sin^{r-s} + b^i_{s,r-s} \cos^s \theta \sin^{r+1-s} \theta \right) d\theta,$$

for $r = 0, 1, \ldots, n$.

Proof. It is straightforward that $\mathcal{F}_n$ is a linearly independent set of $n+1$ functions. Now we shall prove by induction on n that the function $f_n(\rho)$ is a linear combination of the linearly independent set of functions $\mathcal{F}_n$.

Indeed if $n = 1$ by direct computation we have that

$$f_1(\rho) = 2^m \rho^{-2m}[A_0 + \rho A_1],$$

where

$$\begin{aligned} A_0 &= \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \left(a^i_{0,0} \cos\theta + b^i_{0,0} \sin\theta \right) d\theta, \\ A_1 &= \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \left(\cos\theta (a^i_{0,1} \sin\theta + a^i_{1,0} \cos\theta) + \sin\theta (b^i_{0,1} \sin\theta + b^i_{1,0} \cos\theta) \right) d\theta. \end{aligned}$$

So the lemma holds for $n = 1$.

By hypothesis of induction we assume that $f_{n-1}(\rho)$ is a linear combination of functions in $\mathcal{F}_{n-1}$, i.e.

$$f_{n-1}(\rho) = \sum_{r=0}^{n-1} 2^m A_r \rho^{-2m+r},$$

where

$$A_r = \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \sum_{s=0}^{r} \left(a^i_{s,r-s} \cos^{s+1}\theta \sin^{r-s} + b^i_{s,r-s} \cos^s \theta \sin^{r+1-s} \theta \right) d\theta,$$

for $r = 0, 1, \ldots, n-1$.

For n, by a direct integration of the average function we obtain

$$
\begin{aligned}
f_n(\rho) = {} & \frac{2^m}{\rho^{2m}} \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \Big(\cos\theta \sum_{s=0}^{n} \sum_{j=s}^{n} a^i_{s,j-s}\, \rho^j \cos^s\theta \sin^{j-s}\theta \\
& + \sin\theta \sum_{s=0}^{n} \sum_{j=s}^{n} b^i_{s,j-s}\, \rho^j \cos^s\theta \sin^{j-s}\theta \Big)\, d\theta \\
= {} & \frac{2^m}{\rho^{2m}} \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \Big(\sum_{s=0}^{n-1} \sum_{j=s}^{n-1} a^i_{s,j-s} \rho^j \cos^{s+1}\theta \sin^{j-s}\theta \\
& + \sum_{s=0}^{n} a^i_{s,n-i} \rho^n \cos^{s+1}\theta \sin^{n-s}\theta \\
& + \sum_{s=0}^{n-1} \sum_{j=s}^{n-1} b^i_{s,j-s} \rho^j \cos^s\theta \sin^{j-s+1}\theta \\
& + \sum_{s=0}^{n} b^i_{s,n-i} \rho^n \cos^s\theta \sin^{n+1-s}\theta \Big)\, d\theta \\
= {} & f_{n-1}(\rho) + \frac{2^m}{\rho^{2m-n}} \sum_{i=1}^{k} \Big(\int_{\theta_i}^{\theta_{i+1}} \Big(\sum_{s=0}^{n} a^i_{s,n-s} \cos^{s+1}\theta \sin^{n-s}\theta \\
& + \sum_{s=0}^{n} b^i_{s,n-s} \cos^s\theta \sin^{n+1-s}\theta \Big)\, d\theta \\
= {} & f_{n-1}(\rho) + 2^m \rho^{-2m+n} A_n,
\end{aligned}
$$

where

$$
A_n = \sum_{i=1}^{k} \int_{\theta_i}^{\theta_{i+1}} \sum_{s=0}^{n} \Big(a^i_{s,n-s} \cos^{s+1}\theta \sin^{n-s}\theta + b^i_{s,n-s} \cos^s\theta \sin^{n+1-s}\theta \Big)\, d\theta.
$$

This completes the proof of the induction and consequently of the lemma. $\square$

For each $r = 0, 1, \ldots, n$, Lemma 2 assures that the coefficients $a^i_{s,r-s}$ and $b^i_{s,r-s}$ of the vector field $\mathcal{X}_\varepsilon$ which appear in A_r are arbitrarily chosen. So, it follows that the averaging function $f_n(\rho)$ is an arbitrary combinations of functions in the set $\mathcal{F}_n$. Using the *Descartes Theorem*, it follows that $f_n(\rho)$ can have at most n simple zeros, and therefore for $\varepsilon > 0$ sufficiently small the discontinuous polynomial vector field $\mathcal{X}_\varepsilon$ can have at most n limit cycles if $f_n(\rho)$ is not identically zero. This completes the proof of Theorem 1.

3. Some examples

In order to exemplify the result of Theorem 1, we fix $m = 1$ and assume that $\mathbb{R}^2$ splits in two sectors S_1 and S_2 defined by the values $\theta_1 = \pi/2$ and $\theta_2 = 3\pi/2$.

For $n = 1$ we consider the discontinuous piecewise polynomial vector field

$$\mathcal{X}_\varepsilon(x, y) = \Big(-\frac{y}{2}(x^2 + y^2), \frac{x}{2}(x^2 + y^2)\Big) + \begin{cases} \left(\dfrac{2\varepsilon x}{\pi}, \varepsilon y\right) & \text{if } (x, y) \in S_1, \\ (\varepsilon(1 - y), -\varepsilon y) & \text{if } (x, y) \in S_2. \end{cases}$$

Its average function is $f_1(\rho) = (2\rho - 4)/\rho^2$, having a unique positive simple zero, namely 2. Therefore for $\varepsilon > 0$ sufficiently small the discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon$ has one limit cycle $(x(t, \varepsilon), y(t, \varepsilon))$ such that it tends to the circle of radius 2 when $\varepsilon \to 0$.

For $n = 2$ we consider the discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon(x, y)$ defined by

$$\Big(-\frac{y}{2}(x^2 + y^2), \frac{x}{2}(x^2 + y^2)\Big) + \begin{cases} (\varepsilon(1 - x^2 - xy + y^2), -\varepsilon y^2) & \text{if } (x, y) \in S_1, \\ \left(-2\varepsilon y, -\varepsilon y\left(\dfrac{10}{3\pi} + \dfrac{3x}{2}\right)\right) & \text{if } (x, y) \in S_2. \end{cases}$$

Then its average function $f_2(\rho) = (2\rho^2 - 10\rho + 12)/(3\rho^2)$, having two unique positive simple zeros, namely 2 and 3. Therefore for $\varepsilon > 0$ sufficiently small the discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon$ has two limit cycles $(x_i(t, \varepsilon), y_i(t, \varepsilon))$ for $i = 1, 2$ such that they tend to the circles of radius 2 and 3 when $\varepsilon \to 0$.

For $n = 3$ we consider the discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon(x, y)$ defined by

$$\Big(-\frac{y}{2}(x^2 + y^2), \frac{x}{2}(x^2 + y^2)\Big) + \begin{cases} (\varepsilon(2 + 4xy + y^3), -4\varepsilon x) & \text{if } (x, y) \in S_1, \\ \left(6\varepsilon y, \varepsilon\left(-\dfrac{26y}{3\pi} - \dfrac{9xy}{4} - \dfrac{4x^2 y}{3\pi}\right)\right) & \text{if } (x, y) \in S_2. \end{cases}$$

Its average function is $f_3(\rho) = (-\rho^3 + 9\rho^2 - 26\rho + 24)/(3\rho^2)$, having three unique positive simple zeros, namely 2, 3 and 4. Therefore for $\varepsilon > 0$ sufficiently small the discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon$ has three limit cycles $(x_i(t, \varepsilon), y_i(t, \varepsilon))$ for $i = 1, 2, 3$ such that they tend to the circles of radius 2, 3 and 4 when $\varepsilon \to 0$.

4. Appendix: Averaging theory for discontinuous piecewise differential systems

The results stated in this subsection on the averaging theory are valid for discontinuous piecewise vector fields defined in $\mathbb{R}^n$ and are proved in [21], but we shall state them for our discontinuous piecewise polynomial vector field $\mathcal{X}_\varepsilon$ written in polar coordinates as the differential equation (1).

Consider a non-autonomous discontinuous piecewise vector field

$$\frac{d\rho}{d\theta} = \mathcal{X}(\theta, \rho) = \varepsilon F(\theta, \rho) + \varepsilon^2 R(\theta, \rho, \varepsilon),$$

where $\rho \in \mathbb{R}$, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and

$$F(\theta,\rho)=\sum_{i=1}^{k}\chi_{S_i}(\theta)F_i(\theta,\rho), \qquad R(\theta,\rho,\varepsilon)=\sum_{i=1}^{k}\chi_i(\theta)R_i(\theta,c,\varepsilon),$$

where $F_i:\mathbb{S}^1\times D\to\mathbb{R}^2$, $R_i:\mathbb{S}^1\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^2$ for $i=1,\ldots,k$ are continuous functions, 2π-periodic in the variable θ, and D is an open interval of $\mathbb{R}$. Here the S_i are the open intervals (θ_i,θ_{i+1}) for $i=1,\ldots,k$ and $0\le\theta_1<\ldots<\theta_k<2\pi\le\theta_{k+1}=\theta_1+2\pi$. We define

$$D_\rho F(\theta,\rho)=\sum_{i=1}^{k}\chi_{S_i}(\theta,\rho)D_\rho F_i(\theta,\rho).$$

The *average function* $f:D\to\mathbb{R}$ is defined by

$$f(\rho)=\int_0^T F(\theta,\rho)\,d\theta.$$

We recall that if $\rho(\theta,\rho_0)$ is the solution of the vector field $\mathcal{X}(\theta,\rho)$ such that $\rho(0,\rho_0)=\rho_0$, then we have

$$\rho(2\pi,\rho_0)-\rho_0=\varepsilon f(\rho)+\emptyset(\varepsilon^2).$$

So for $\varepsilon>0$ sufficiently small the simple zeros of the average function $f(\rho)$ provides limit cycles of the vector field $\mathcal{X}(\theta,\rho)$.

In the next result we present a version of the averaging theory for discontinuous piecewise vector fields, that is proved in [21], adapted to our differential equation (1). We note that in [21] the averaging theory uses that the Brouwer degree of a function f in a neighborhood of a zero $\overline{\rho}$ of the function $f(\rho)$ is non-zero, while here we substitute this condition saying that the zero $\overline{\rho}$ is simple (i.e. $\dfrac{df}{d\rho}(\overline{\rho})\neq 0$), because this last condition implies that the mentioned Brouwer degree is non-zero. See for more details [7, 25].

Theorem 3. *Assume that the following conditions hold for the discontinuous piecewise vector field $\mathcal{X}(\theta,\rho)$.*

(i) *For $i=1,\ldots,k$ the functions $F_i(\theta,\rho)$ and $R_i(\theta,\rho)$ are locally Lipschitz with respect to ρ, and 2π-periodic with respect to θ.*
(ii) *Let $\overline{\rho}\in D$ be a simple zero of the average function $f(\rho)$.*

Then for $\varepsilon>0$ sufficiently small, there exists a 2π-periodic solution $\rho(\theta,\varepsilon)$ of the vector field $\mathcal{X}(\theta,\rho)$ such that $\rho(0,\varepsilon)\to\overline{\rho}$ as $\varepsilon\to 0$.

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References

[1] A.A. Andronov, *Les cycles limites de Poincaré et la théorie des oscillations auto–entretenues*, C. R. Acad. Sci. Paris **189** (1929), 559–561.

[2] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[3] J.C. Artés, J. Llibre, J.C. Medrado and M.A. Teixeira, *Piecewise linear differential systems with two real saddles*, Math. Comp. Sim. **95** (2013), 13–22.

[4] I.S. Berezin and N.P. Zhidkov, *Computing Methods*, Volume II, Pergamon Press, Oxford, 1964.

[5] M. Di Bernardo, C. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise–smooth dynamical systems: theory and applications*, Appl. Math. Sci. Vol. **163**, Springer–Verlag, New York, 2007.

[6] D.C. Braga and L.F. Mello, *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane*, Nonlinear Dynamics **73** (2013), 1283–1288.

[7] F. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.

[8] A. Buică, J. Giné, J. Llibre, *Bifurcation of limit cycles from a polynomial degenerate center*, Advanced Nonlinear Studies **10** (2010), 597–609.

[9] C. Buzzi, C. Pessoa and J. Torregrosa, *Piecewise linear perturbations of a linear center*, Discrete Continuous Dyn. Syst. **9** (2013), 3915–3936.

[10] S.N. Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge Univ. Press, 1994.

[11] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.

[12] E. Freire, E. Ponce and F. Torres, *Canonical Discontinuous Planar Piecewise Linear Systems*, SIAM J. Applied Dynamical Systems **11** (2012), 181–211.

[13] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynamics **78** (2014), 251–263.

[14] F. Giannakopoulos and K. Pliete, *Planar systems of piecewise linear differential equations with a line of discontinuity*, Nonlinearity **14** (2001), 1611–1632.

[15] M. Han and W. Zhang, *On Hopf bifurcation in non–smooth planar systems*, J. of Differential Equations **248** (2010), 2399–2416.

[16] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Discrete and Continuous Dynamical Systems-A **32** (2012), 2147–2164.

[17] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems of node–node types*, J. Math. Anal. Appl. **411** (2013), 340–353.

[18] S.M. Huan and X.S. Yang, *Existence of limit cycles in general planar piecewise linear systems of saddle–saddle dynamics*, Nonlinear Anal. **92** (2013), 82–95.

[19] I.D. Iliev, *The number of limit cycles due to polynomial perturbations of the harmonic oscillator*, Mat. Proc. Camb. Phil. Soc. **127** (1999), 317–322.

[20] A. Liénard, *Etude des oscillations entretenues*, Rev. Générale de l'Electricité **23** (1928), 901–912.

[21] J. Llibre, A.C. Mereu and D.D. Novaes, Âveraging theory for discontinuous piecewise differential systems, J. Differential Equations **258** (2015), 4007–4032.

[22] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Analysis Series B: Real World Applications **14** (2013), 2002–2012.
[23] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dynamics of Continuous, Discrete and Impulsive Systems, Serie B **19** (2012), 325–335.
[24] J. Llibre, M.A. Teixeira and J. Torregrosa, *Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation*, Int. J. Bifurcation and Chaos **23** (2013), 1350066, pp. 10.
[25] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.
[26] O. Makarenkov and J.S.W. Lamb, *Dynamics and bifurcations of nonsmooth systems: A survey*, Physica D **241** (2012), 1826–1844.
[27] H. Poincaré, *Mémoire sur les courbes définies par une équation differentielle I, II*, J. Math. Pures Appl. **7** (1881), 375–422; **8** (1882), 251–96; Sur les courbes définies par les équation differentielles III, IV **1** (1885), 167–244; **2** (1886), 155–217.
[28] van der Pol, *On relaxation–oscillations*, Phil. Mag. **2** (1926), 978–992.
[29] L.S. Pontrjagin, *Über Autoschwingungssysteme, die den hamiltonschen nahe liegen*, Physikalische Zeitschrift der Sowjetunion **6** (1934), 25–28.
[30] S. Shui, X. Zhang and J. Li, *The qualitative analysis of a class of planar Filippov systems*, Nonlinear Anal. **73** (2010), 1277–1288.
[31] D.J.W. Simpson, *Bifurcations in Piecewise–Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol **69**, World Scientific, Singapore, 2010.
[32] Ye Yanqian, *Theory of Limit Cycles*, Translations of Math. Monographs **66**, Amer. Math. Soc., Providence, RI, 1986.

[1] Departamento de Matemática, Faculdade de Ciências, UNESP, Av. Eng. Luiz Edmundo Carrijo Coube 14-01, CEP 17033-360, Bauru, SP, Brazil.

E-mail address: `tcarvalho@fc.unesp.br`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[3] Institute of Mathematics and Statistics of Federal University of Goiás, Avenida Esperança s/n, Campus Samambaia, CEP 74690-900, Goiânia, Goiás, Brazil.

E-mail address: `djtonon@ufg.br`