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Magdalena CAUBERGH

Promotor : Prof. dr. Freddy Dumortier
Co-promotor : Prof. dr. Robert Roussarie

LIMBURGS UNIVERSITAIR CENTRUM
PARTNER IN DE UNIVERSITEIT LIMBURG

Preface

The subject of the thesis concerns the cyclicity and the bifurcations of limit cycles in C^∞ and analytic families $(X_\lambda)_\lambda$ of planar vector fields near vector fields of center type.

As the parameter λ varies, changes may occur in the phase portraits of the vector fields X_λ. These changes are called bifurcations and the parameter values λ^0, at which such a bifurcation occurs, are called bifurcation values; the vector field X_{λ^0} is called the bifurcation vector field. In this thesis, the bifurcation vector field X_{λ^0} is of center type, meaning that its phase portrait contains an annulus or a punctured disc of non-isolated periodic orbits; typical examples of such vector fields are Hamiltonian vector fields.

Recall that an isolated periodic orbit of X_λ is called a limit cycle. Our analysis is of a local kind, in the sense that we study bifurcations of limit cycles in the neighbourhood of a so-called limit periodic set Γ. Attention is focussed on the maximal possible number of limit cycles of X_λ, that can arise in the neighbourhood of Γ, after small perturbations of the parameter λ near λ^0; this number is referred to as the cyclicity of X_λ at $\left(\Gamma, \lambda^0\right)$. The limit periodic sets Γ, that are considered in this thesis, are (regular) periodic orbits, non-degenerate elliptic points and 2-saddle cycles.

The study of bifurcations of limit cycles and their cyclicity is motivated by Hilbert's sixteenth problem, that asks for a bound H_n for the maximum number of limit cycles (and their relative positions) in polynomial vector fields in the plane of degree n, only depending on the degree n. Although Hilbert's sixteenth problem is of a global character, it is known that a solution to all local problems induces the existence of a finite H_n (see [R98])

Traditionally, the study of limit cycles of planar vector fields $(X_\lambda)_\lambda$ near limit periodic sets, as the ones we consider, is replaced by the study of isolated fixed points of associated 1-dimensional Poincaré-maps $(P_\lambda)_\lambda$, or equivalently, by the study of isolated zeroes or so-called displacement maps $(\delta_\lambda)_\lambda$, defined by $\delta_\lambda = P_\lambda - Id$. In such a way, configurations of isolated zeroes of δ_λ correspond to configurations of limit cycles of X_λ.

In the study of stable bifurcation diagrams near a non-degenerate elliptic singularity, people often use techniques such as normal forms or Lyapunov quantities.

To study the cyclicity near a periodic orbit or a non-degenerate elliptic singularity, when the bifurcation vector field X_{λ^0} is of center type, there is the well-known technique of computing Abelian integrals (the so-called Melnikov functions) in 1-parameter families, and the technique of the Bautin ideal in multi-parameter families. For instance, the first order Melnikov function is the coefficient in the linear approximation of the displacement map δ_λ, with respect to λ. The technique of the Bautin ideal is based on a special division of the displacement map; for 1-parameter families this technique reduces to the technique of computing Melnikov functions. In the literature, there exist algorithms to compute Melnikov functions, while the Bautin ideal is a very powerful theoretical technique, that often, in practice, is too difficult to be computed.

In this thesis, we focus on three problems, that briefly can be described as follows. The first problem deals with stable bifurcation diagrams of limit cycles near centers, where attention is focused on uniform results as well in phase plane as well as in parameter space.

The second problem is the investigation of how 1-parameter techniques, such as the computation of Melnikov functions, can be used in multi-parameter families, to compute its cyclicity near centers.

The third problem deals with families $\big(X_{(\nu,\varepsilon)}\big)$ of planar vector fields that unfold a Hamiltonian vector field for $\varepsilon = 0$, where ε is a 1-parameter; it is the investigation whether results on linear approximations I_ν of the displacement map $\delta_{(\nu,\varepsilon)}$, with respect to ε (such as the first order Melnikov function), can be transferred to valuable results on the bifurcation diagram of limit cycles and the cyclicity. Let us now describe these problems in more detail.

Related to the first problem, a well-known example of a stable bifurcation pattern is the Andronov-Hopf bifurcation in the neighbourhood of a non-degenerate elliptic singularity (i.e. with pure imaginary eigenvalues), the so-called Hopf singularity. By the implicit function theorem, it follows that under small perturbations of the vector field, the singularity persists and no new singularities are created. However, it is possible that the stability type of the singularity changes when subjected to perturbations, and then this change is usually accompanied with either the appearance or disappearance of a small limit cycle encircling the singularity. This important well-known bifurcation phenomenon is called the Andronov-Hopf bifurcation.

Generalisations of the Andronov-Hopf bifurcation, giving rise to multiple limit cycles, are called generalised Hopf bifurcations or Hopf-Takens bifurcations. A precise study of generic generalised Hopf bifurcations is done in [T], by way of normal forms, when no centers occur.

Perturbations from centers naturally show up in many problems and one constantly has to consider Hopf-Takens bifurcations that perturb from a center. In this thesis, we link the different techniques that are used in the study of a Hopf-singularity, surrounded by non-isolated periodic orbits: normal forms, Lyapunov quantities and Melnikov functions.

In the study of bifurcation diagrams of a family $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, there appears besides a 1-parameter ε, inducing centers for $\varepsilon = 0$, also an external parameter ν, that controls bifurcations from these centers. If the centers are exclusively situated at $\varepsilon = 0$, then we speak of a 'regular hypersurface of centers'. In case of a regular hypersurface of centers, we prove a result for each of these techniques, indicating precisely which verifications have to be made in order to guarantee the presence of a generic Hopf-Takens bifurcation, on a uniform domain, both in phase plane as in parameter space (i.e. a domain that does not shrink with the bifurcation value $\varepsilon \downarrow 0$).

Finally, we consider one more example in which centers are generated by a 2-parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$. In this example, besides the Hopf bifurcation also another type of bifurcation shows up. A limit cycle disappears through the boundary of the domain; this bifurcation is called a boundary bifurcation.

The second problem in this thesis deals with analytic families of planar vector fields $(X_\lambda)_\lambda$, investigating methods to detect the cyclicity in the multi-parameter family $(X_\lambda)_\lambda$ at a non-isolated closed orbit Γ, by means of 1-parameter subfamilies. In [R00], using the desingularisation theory of Hironaka, Roussarie constructed a polynomial curve $\lambda(\varepsilon)$ in parameter space, such that the first non-identical zero Melnikov function of the induced 1-parameter subfamily $\left(X_{\lambda(\varepsilon)}\right)_\varepsilon$, can be used to bound the cyclicity of the multi-parameter family. This curve $\lambda(\varepsilon)$ is called a curve of maximal index (mic). In the spirit of this result, we prove, using the theory of analytic geometry, that the multi-parameter problem can be reduced to a 1-parameter one, in the sense that there exist analytic curves in parameter space along which the maximal cyclicity can be attained. In that case one speaks about a maximal cyclicity curve (mcc) if only the number is considered and of a maximal multiplicity curve (mmc) if the multiplicity is also taken into consideration. In view of obtaining efficient algorithms for detecting the cyclicity, we investigate whether such mcc, mmc and mic can be algebraic or even linear depending on certain general properties of the families or of their associated Bautin ideal. In any case by well chosen examples we show that prudence is appropriate.

In most examples encountered in the literature, nearby vector fields, with maximal cyclicity (respectively multiplicity) are structurally stable and hence occur in open subanalytic sets of the parameter space. In case the stratum of maximal cyclicity has a non-empty interior adhering at λ^0, we show that there always exists an algebraic mcc (respectively mmc) ζ, in case the analytic family of planar vector fields has a stratum of maximal cyclicity (respectively multiplicity) with non-empty interior at λ^0. In particular, in that case there exists a 'cone of mcc's (respectively mmc's) surrounding ζ'.

For certain specific examples, we also discuss related questions such as the existence of minimal detectibility and conic degree of maximal cyclicity (respectively multiplicity).

The third problem deals with C^∞ families $\left(X_{(\nu,\varepsilon)}\right)$ of planar vector fields, that unfold a Hamiltonian vector field X_H for $\varepsilon = 0$, where ε is a 1-dimensional parameter. It asks how results on linear approximations I_ν of the displacement map $\delta_{(\nu,\varepsilon)}$, with respect to ε, can be transferred to valuable results on the bifurcation diagram of limit cycles and the cyclicity in $\left(X_{(\nu,\varepsilon)}\right)$. If $\left(\upsilon_{(\nu,\varepsilon)}\right)$ is the C^∞ family of dual 1-forms associated to the family $\left(X_{(\nu,\varepsilon)}\right)$, then it is well-known that I_ν can be computed by integration of the first order approximation of $\upsilon_{(\nu,\varepsilon)}$ with respect to ε along the level curves $\Gamma_x \subset \{H = x\}$ of the Hamiltonian H : if

$$\upsilon_{(\nu,\varepsilon)} = \mathrm{d}H + \varepsilon \bar{\upsilon}_\nu + o(\varepsilon), \varepsilon \to 0,$$

then

$$I_\nu(x) = -\int_{\Gamma_x} \bar{\upsilon}_\nu,$$

where Γ_x is oriented by the vector field X_H. Therefore, we refer to I_ν as the related Abelian integral of the family $\left(X_{(\nu,\varepsilon)}\right)$.

In case Γ is a periodic orbit or a non-degenerate elliptic singularity, then it is well-known that results on configurations of isolated zeroes of the related Abelian integral I_ν can be transferred to results on configurations of limit cycles of the family in a trivial way, at least if the Abelian integral represents an elementary catastrophe.

In dealing with a k-saddle cycle Γ (i.e. a hyperbolic polycycle with k saddle-type singular points), the transfer of the results on the related Abelian integral I_ν is no longer obvious. The difficulties are due to the fact that the displacement map is not C^∞ at the saddle points, unlike the case when Γ is a periodic orbit or a non-degenerate singular point.

In dealing with a 1-saddle cycle or a so-called saddle loop, it is known from [Mar], that under certain genericity conditions on the Abelian integral I_ν, the configuration of limit cycles of $X_{(\nu,\varepsilon)}$, for ε close to 0, is completely analoguous to the configuration of zeroes of I_ν.

In general, unlike the case of the regular periodic orbit or the saddle loop, the bifurcation diagram of limit cycles near a k-saddle cycle is no longer trivial in the ε-direction. The bifurcation diagram of a 2-saddle cycle is studied in [DRR], and more generally, the generic k-parameter unfoldings of k-saddle cycles are studied in [Mo]. Using these results, it is proven in [DR], that the Abelian integral is a very bad approximation of the displacement map as soon as the unfolding breaks more than one connection: almost all the limit cycles cannot be traced by the Abelian integral.

It is even not obvious whether it is possible to transfer results on the Abelian integral to obtain valuable results on the cyclicity along the 2-saddle cycle. Even in case the unfolding keeps one connection of the 2-saddle cycle unbroken, the transfer does not work out in a trivial way, unlike one could expect by the known results on the saddle loop.

In [DR], it is proven that there exist generic unfoldings of 2-saddle cycles leaving one connection unbroken, for which the cyclicity is 4, while the related Abelian integral I_ν is of codimension 3, and hence can produce at most 3 zeroes. As a consequence, in that case, one limit cycle is not covered by a zero of the related Abelian integral. Such a limit cycle is called an alien limit cycle.

However, the problem of transfer can be dealt with. From [DR], it is known that the Abelian integral I_ν provides a finite upperbound for the cyclicity, if it is of finite codimension. It is interesting to notice that the upperbound in this finite cyclicity result, is strictly bigger than the maximal possible zeroes of the related Abelian integral. Therefore, it is possible that, in general, the family creates more alien limit cycles near the considered 2-saddle cycle.

The thesis is organised as follows.

Chapter 1 recalls the techniques that are used in the study of the bifurcation of limit cycles and the cyclicity near centers.

In chapter 2, we present a complete and clear reference work on Hopf-Takens bifurcations (generic and near centers), aiming, in applications, at obtaining accurate results based on a minimal amount of verification. In chapter 3, we apply these results to the study of bifurcations of small-amplitude limit cycles in families originating from classical and generalized Liénard equations. The simplicity of the Liénard family is used to illustrate the advantages of the approach based on Bautin ideals. The Bautin ideal is generated by a set of Lyapunov quantities. Attention goes to the local division of a family of displacement maps, the presence of Hopf-Takens bifurcations, and the cyclicity.

In chapter 4, we examine the use of 1-parameter techniques in analytic multi-parameter families.

Chapter 5 deals with unfoldings of a 2-saddle cycle, leaving one connection unbroken, extending the results of [DR] and indicating new problems that show up in generalising this study. Special interest goes to the existence of alien limit cycles. The existence of alien limit cycles implies that knowlegde of the linear approximation I_ν, with respect to $\varepsilon = 0$, is not sufficient to transfer results on zeroes of I_ν in a trivial way to valid results on limit cycles, arbitrarily close to Γ, of the unfolding $X_{(\nu,\varepsilon)}$ (for $\lambda = (\nu, \varepsilon)$ near λ^0). The study in chapter 5 gives rise to the following conjecture: 'A generic unfolding of the 2-saddle cycle, leaving one connection unbroken, can produce $3k$ (respectively $3k - 1$) limit cycles, while the related Abelian integral is of codimension $2k + 1$ (respectively $2k$)'. This conjecture would imply the existence of at least $k - 1$ alien limit cycles. Furthermore, we prove that a particular subfamily of the 2-saddle cycle, leaving one connection unbroken and in which the saddles remain linear at the bifurcation, can produce at least $k - 2$ alien limit cycles, if the related Abelian integral is of codimension k.

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