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Separatrix skeleton and limit cycles in some 1-parameter families of planar vector fields

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Abstract— We consider polynomial vector fields X_m^k of degree $4k+1$ given by $\dot{x} = y^3 - x^{2k+1}, \dot{y} = -x + my^{4k+1}$, $x, y \in \mathbb{R}$, where $m \in \mathbb{R}$ is a parameter. For any $k \geq 1$ we analyze the bifurcation of the separatrix skeleton of $(X_m^k)_{m>0}$ analytically. For $k = 1$ the bifurcation diagram of global phase portraits is deduced. Related to it, we address Hilbert's 16th Problem and the nilpotent Center-Focus Problem, restricting both problems to this 1-parameter family. This contribution summarizes the author's main results published in *Separatrix skeleton for some 1-parameter families of planar vector fields* in J Differ Equations **259** (2015). Additionally, we discuss the separatrix bifurcation for a generalization of the 1-parameter family $(X_m^k)_{m>0}$.

Keywords: planar vector field, separatrix skeleton, limit cycle, Hilbert's 16th problem, nilpotent center problem.

1 Introduction

We consider the 1-parameter family of planar vector fields

$$X_m^k \leftrightarrow \dot{x} = y^3 - x^{2k+1}, \dot{y} = -x + my^{4k+1}, \ (x, y) \in \mathbb{R}^2,$$

where $k \geq 1$ is an arbitrary but fixed integer and m a real parameter. During a conference on stability for differential equations held in Florence in 1985, Bacciotti asked for the stability type of the nilpotent singularity of X_m^k at the origin, and how its change of stability relates to the appearance of a polycycle. Shortly after that conference Galeotti and Gori presented a work in [5] on the stability type of the origin of X_m^k. More recently, Gasull, García and Giacomini reconsidered that problem using generalized Lyapunov focus quantities in [6]; furthermore they present a complete and rather technical study on limit cycles in the case that $k = 1$. In [2] the author completes the study of global phase portraits of X_m^1, by an analysis of the separatrix skeleton of X_m^k in function of m, for all $k \geq 1$ (see Sections 2 and 3). Besides, this study is used to exclude centers for X_m^k and to prove that the Hilbert number for $(X_m^k)_{m\in\mathbb{R}}$ is finite (see Sections 4 and 5). Both Hilbert's 16th Problem and the Center-Focus Problem are longstanding challenges from the beginning of the 20th century, and so-far a complete solution for them is not yet known beyond linear and quadratic differential equations respectively.

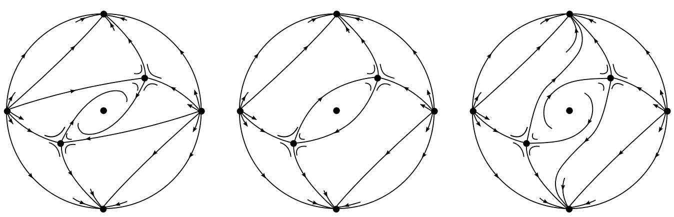
Here we recall the main results from [2] and we provide as well with outlines for their proofs, counting on a whole arsenal of local and global machinery from Qualitative Theory of Differential Equations. Additionally, we discuss some generalizations to replace X_m^k.

2 Separatrix skeleton for $k \geq 1$

2.1 Definitions and main result

Let X be a continuous planar vector field having only isolated singularities. An orbit Γ of X is called *separatrix* if it is homeomorphic to $\mathbb{R}$ and for each neighborhood $\mathcal{N}$ of Γ there exists $q \in \mathcal{N}$ such that $\alpha(q) \neq \alpha(\Gamma)$ or $\omega(q) \neq \omega(\Gamma)$. The closure of the union of separatrices is called the *separatrix skeleton of* X. In next theorem we present the key result from [2].

THEOREM 1 ([2]) *X_m^k undergoes a unique separatrix bifurcation for increasing $m > 0$, giving subsequently rise to the following three separatrix skeletons:*



In the subsequent subsections we prove the existence of a unique parameter value $m = m_C(k)$ at which X_m^k exhibits a 2-saddle cycle. For that aim we use a parameter dependent coordinate transformation that brings the family $(X_m^k)_{m>0}$ into a semi-complete family of indefinitely rotated vector fields.

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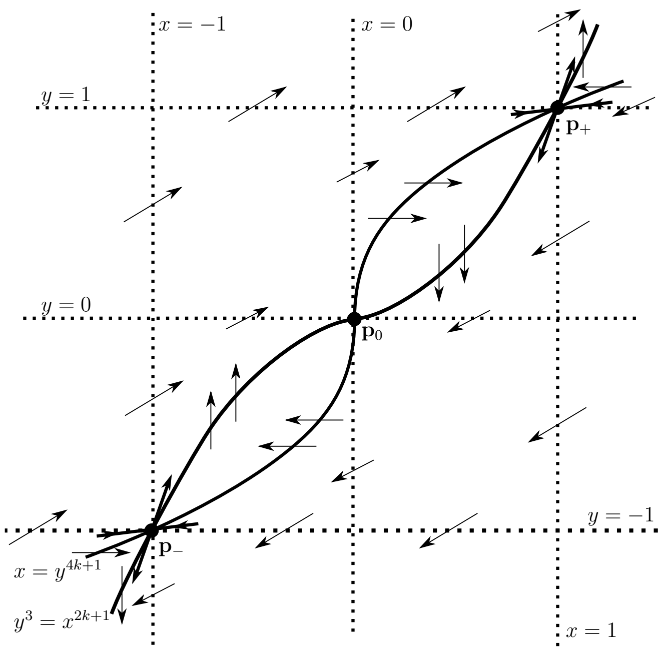
2.2 Preliminaries

We perform a parameter dependent rescaling of variables and time to each of the X_m^k, and obtain the following topologically equivalent vector fields $X_m^{k,R}, m > 0$

$$X_m^{k,R} \leftrightarrow \bar{x}' = \bar{y}^3 - \bar{x}^{2k+1},\ \bar{y}' = m^{\frac{1}{k+1}}(-\bar{x} + \bar{y}^{4k+1}).$$

Let $m_S(k) \equiv (2k+1)!!/(4k+1)!!!!$. Then it follows from the study in [6] that for $m \neq m_S(k)$ the vector field $X_m^{k,R}$ has a nilpotent focus at the origin, which is attracting for $m < m_S(k)$ and repelling for $m > m_S(k)$. Besides, for $m > 0$ the vector field $X_m^{k,R}$ has a fixed pair of hyperbolic saddles at $\mathbf{p}_\pm = (\pm 1, \pm 1)$. Clearly the flow of $X_m^{k,R}$ as well as X_m^k are invariant with respect to $(t, \bar{x}, \bar{y}) \mapsto (t, -\bar{x}, -\bar{y})$. As a consequence, if there is a connection between the saddles $\mathbf{p}_+$ and $\mathbf{p}_-$, it immediately follows that a 2-saddle cycle exists. In particular, polycycles necessarily are 2-saddle cycles.

Elementary calculations show that the directional vector field for $X_m^{k,R}$ is given by next figure.



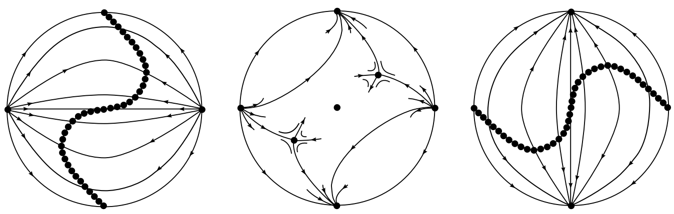
In particular in a neighborhood of $\mathbf{p}_\pm$ the separatrices at $\mathbf{p}_\pm$ are localized. Moreover it is straightforward that both saddles $\mathbf{p}_\pm$ have one unbounded stable and unstable separatrix inside $\{|x| > 1, |y| > 1, xy > 0\}$.

Denote by $\Gamma_+(m)$ (resp. $\Gamma_-(m)$) the stable (resp. unstable) separatrix for $X_m^{k,R}$ at $\mathbf{p}_+$ (resp. $\mathbf{p}_-$) having a non-empty intersection with the cube $\mathcal{C} \equiv [-1,1] \times [-1,1]$.

Proposition 2 *Let $k \geq 1, m > 0$. Polycycles and limit cycles of $X_m^{k,R}$ are contained in the cube $\mathcal{C}$. If both $\Gamma_\pm(m)$ are bounded, then $\Gamma_+(m) = \Gamma_-(m)$, and a 2-saddle cycle is formed for $X_m^{k,R}$. If $\Gamma_\pm(m)$ is unbounded, then $\Gamma_\mp(m) \subset \mathcal{C}$.*

In Section 2.3 we extend the family of vector fields $X_m^{k,R}$ with $m \in (0, \infty)$ in a natural way to $m \in [0, \infty]$.

Furthermore by Poincaré compactification all these polynomial vector fields can be extended analytically to a compact $\mathcal{D}^1$, the so-called Poincaré disc; the compactified vector fields are denoted by $\hat{X}_m^{k,R}$. From the analysis of the critical points on the boundary of $\mathcal{D}^1$ we obtain the asymptotic behavior of trajectories that become unbounded, as illustrated in next figure.



$m = 0$ $\quad m > 0$ $\quad m = \infty$

In particular the behavior near infinity is obtained in a uniform way (i.e. outside a fixed compact set, which does not change when m is varied). This is important when replacing the study of global phase portraits of $X_m^{k,R}$ by the study of bifurcations inside $(X_m^{k,R})_{m>0}$. In this way it can control the movement of the separatrices in the global plane for all $m > 0$ in Section 2.4, and rule out limit cycles escaping to infinity (so-called large amplitude limit cycles) and localize the global absence problem of limit cycles for large m in Section 5.3.

2.3 At most one separatrix bifurcation

In this section we first recall the definition of a semi-complete family of rotated vector fields and two principles these families obey concerning the non-intersection of separatrices and the splitting of hyperbolic saddle connections. Next we observe that $X_m^{k,R}$ is a semi-complete family of indefinitely rotated vector fields, which thus implies the existence of at most one parameter value $m = m_C(k)$ with a connection between $\mathbf{p}_-$ and $\mathbf{p}_+$.

Definition 3 Let $E \subset \mathbb{R}^2$ be connected, $I \subset \mathbb{R}$ an interval and $f = (f_1, f_2) : E \times I \to \mathbb{R}^2$, $G : E \to \mathbb{R}$ analytic functions such that $G^{-1}(0)$ does not contain any cycle of $X_\lambda \leftrightarrow \dot{x} = f(x, \lambda)$. Then, $(X_\lambda)_{\lambda \in I}$ is said to be a

1. semi-complete family of positively (resp. negatively) rotated vector fields (mod $G = 0$) on E if $(f_1 \frac{\partial f_2}{\partial \lambda} - f_2 \frac{\partial f_1}{\partial \lambda})(x, \lambda) > 0$ (resp. < 0) at all $(x, \lambda) \in E \times I$ for which $f(x, \lambda) G(x) \neq 0$ and the singularities of X_λ do not move with $\lambda \in I$.

2. semi-complete family of indefinitely rotated vector fields (mod $G = 0$), if $(X_\lambda)_{\lambda \in I}$ is a semi-complete family of positively or negatively rotated vector fields on any connected component C of $E \setminus G^{-1}(0)$.

THEOREM 4 ([8]) *Assume that $(X_\lambda)_{\lambda \in I}$ is an analytic semi-complete family of positively rotated vector fields.*

1. *If $S(\lambda)$ is a separatrix at a hyperbolic saddle of $(X_\lambda)_{\lambda \in I}$, then it follows that $S(\lambda_1) \cap S(\lambda_2) = \emptyset$ for $\lambda_1 \neq \lambda_2$. Furthermore the tangent line to $S(\lambda)$ rotates monotonically in the positive sense as λ increases.*
2. *Assume that $S^{\pm}(\lambda)$ are separatrices at the hyperbolic saddles $p_{\pm}$ of $(X_\lambda), \lambda \in I$, and that there is a saddle connection at $\lambda = \lambda_0$, i.e. $S^+(\lambda_0) = S^-(\lambda_0)$. Then, as λ varies from λ_0, the saddle connection splits and if Σ is a smooth curve transverse to $S^+(\lambda_0)$, the separatrices $S^+(\lambda)$ and $S^-(\lambda)$ move in opposite directions along Σ as λ increases.*

Returning to $X_m^{k,R}, m > 0$, we let $G_k : \mathbb{R}^2 \to \mathbb{R}$ be defined by

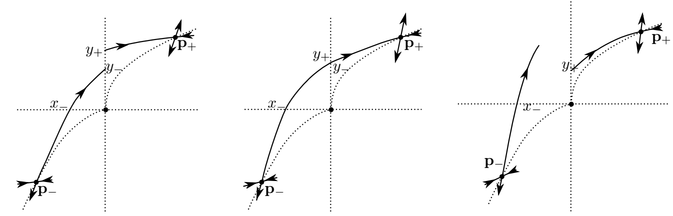
$$G_k(\bar{x}, \bar{y}) = (\bar{y}^3 - \bar{x}^{2k+1})(\bar{y}^{4k+1} - \bar{x}). \tag{1}$$

Hence, the 0-level set $G_k^{-1}(0)$ determines the 0-isoclines for $X_m^{k,R}, m > 0$. Then one can easily check that $(X_m^{k,R})_{m \geq 0}$ is a semi-complete family of indefinitely rotated vector fields (mod $G_k = 0$), that is positively rotated in $G_k^{-1}[0, \infty)$ and negatively rotated in $G_k^{-1}(-\infty, 0]$.

2.4 Existence of separatrix bifurcation

We claim the existence of m_0 and m_∞ for which $\Gamma_+(m_0)$ and $\Gamma_-(m_\infty)$ are unbounded.

Assuming that our claim is true, it follows by Proposition 2 that the relative positions of $\Gamma_\pm(m)$ are opposite in cases $m = m_0$ and $m = m_\infty$. Furthermore the continuous dependence on the parameter then implies the existence of $m_0 < m < m_\infty$ for which $X_m^{k,R}$ has a connection between $\mathbf{p}_+$ and $\mathbf{p}_-$. This idea is illustrated in next figure.



m sufficiently small $\quad m = m_C(k) \quad$ m sufficiently large

Recall that by the rotated property this value m_C is unique.

To prove our claim we analyze the flow of $X_m^{k,R}$ for arbitrarily small $m > 0$ (resp. large $m > 0$), relying on the limiting vector field for $m \to 0$ (resp. $m \to \infty$). The behavior of $X_m^{k,R}$ for arbitrarily large m can be obtained by introducing the new parameter variable $\eta = 1/m$, rescaling time, and taking the limit for $\eta \to 0$. For $\eta > 0$ the vector field $Y_\eta^{k,R}$ thus obtained is given by

$$Y_\eta^{k,R} \leftrightarrow \bar{x}' = \eta^{1/(k+1)}(\bar{y}^3 - \bar{x}^{2k+1}),\ \bar{y}' = -\bar{x} + \bar{y}^{4k+1}.$$

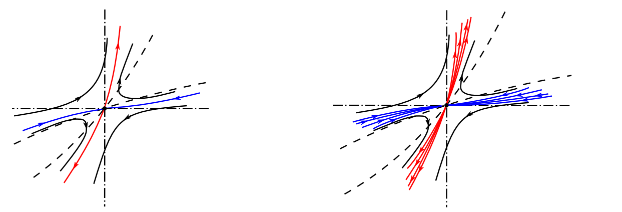
For $m > 0$ the phase portraits of $X_m^{k,R}$ and $Y_{1/m}^{k,R}$ are identical. Clearly, the families of vector fields $X_m^{k,R}, m > 0$ and $Y_\eta^{k,R}, \eta > 0$ extend analytically to $m = 0$ and $\eta = 0$ respectively. The limiting vector field $X_0^{k,R}$ (resp. $Y_0^{k,R}$) exhibits a horizontal (resp. vertical) strip flow with a curve full of singularities. By continuous dependence on initial conditions and parameter, Proposition 2 and the monotonicity principle of separatrix intersections for rotated vector fields, our claim follows.

2.5 Generalizations

Here we provide with a more abstract setting in which the bifurcation result from Theorem 1 remains valid under some genericity condition. Besides we consider a particular case in which the genericity condition is violated though the bifurcation result still partially holds true. For arbitrary fixed integers $k \geq l \geq 1$, we consider the analytic family of vector fields

$$Z_m^{k,l} \leftrightarrow \dot{x} = y^{2l+1} - f(x), \dot{y} = m(-x + g(y)),$$

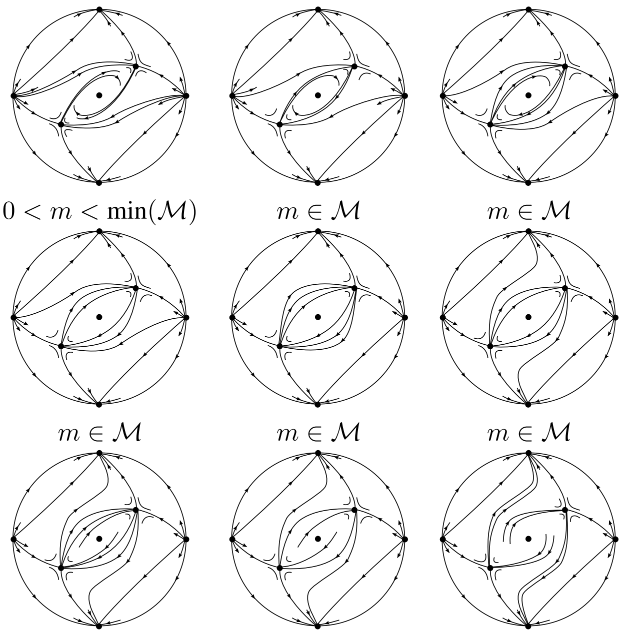
for analytic functions $f, g : \mathbb{R} \to \mathbb{R}$ that are strictly increasing, convex and odd (i.e. $f(-x) = -f(x)$ and $g(-x) = -g(x)$) with jets $j_{2k+1}f_0(x) = x^{2k+1}$ and $j_{4k+1}g_0(y) = y^{4k+1}$. Then $Z_m^{k,l}$ has exactly 3 singularities (independent of m): a nilpotent singularity at the origin and a pair of symmetric singularities $\mathbf{p}_\pm$. Furthermore each $\mathbf{p}_\pm$ has exactly four hyperbolic sectors as in the case of $X_m^{k,R}$, however the singularities are not necessarily elementary neither topological saddles. The hyperbolic sectors can be alternated by repelling and attracting sectors (see [4]). In next figure we present (a) the topological saddle as found for X_m^k and (b) an example of a non-elementary singularity at $\mathbf{p}_+ = (p_{+,1}, p_{+,2})$; both are drawn, in relation to the isoclines and the lines $\{x = p_{+,1}\}$ and $\{y = p_{+,2}\}$.



(a)Topological saddle $\quad$ (b) a non-elementary singularity

In case that $\mathbf{p}_+$ is a topological saddle, Theorem 1 with $X_m^{k,R}$ replaced by $Z_m^{k,l}$ is obtained reasoning in the same way.

In case of a non-elementary singularity at $\mathbf{p}_+$, then using a similar technique as in [3], there still can be found a connected set $\mathcal{M} \subset \mathbb{R}$ of parameters with a connection between $\mathbf{p}_-$ and $\mathbf{p}_+$. However, in that case, the maximal connected invariant set containing $\mathbf{p}_+$ can change when varying m in $\mathcal{M}$. For instance, let us assume that $\mathbf{p}_+$ is of the topological type sketched in (b) for all $m > 0$. Then for $0 < m < \min(\mathcal{M})$ (resp. $m > \max(\mathcal{M})$), there is a unique separatrix skeleton, that is as the one for X_m^k with $0 < m < m_C(k)$ (resp. $m > m_C(k)$), but with the attracting (resp. repelling) separatrix replaced by an attracting (resp. repelling) sector; in between, i.e. for $m \in \mathcal{M}$, there are 7 possible separatrix skeletons as illustrated below.



$0 < m < \min(\mathcal{M})$ | $m \in \mathcal{M}$ | $m \in \mathcal{M}$
$m \in \mathcal{M}$ | $m \in \mathcal{M}$ | $m \in \mathcal{M}$
$m \in \mathcal{M}$ | $m \in \mathcal{M}$ | $m > \max(\mathcal{M})$

3 Bifurcation of phase portraits for $k = 1$

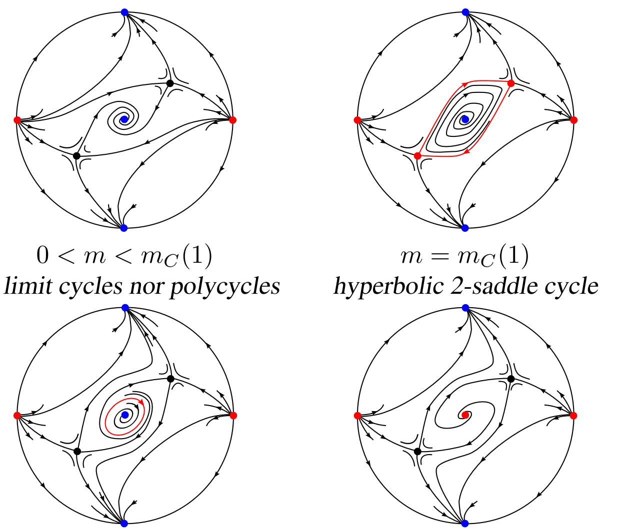
To determine the phase portraits of X_m^k, we rely on the Theorem of Markus, Neumann and Peixoto. It says that continuous planar vector fields with only isolated singularities topologically are uniquely determined by their so-called *completed separatrix skeleton* whose definition is given below.

A *limit cycle* is a periodic orbit γ that is isolated in the Hausdorff sense. Then, according to the definition given in Section 2.1 a limit cycle is not a separatrix and it is not included in the separatrix skeleton. Furthermore, topological sinks and sources are considered as degenerate limit cycles and therefore not included in the separatrix skeleton.

The union of the separatrix skeleton, limit cycles and topological sinks and sources of X is called the *extended separatrix skeleton of* X. Maximal connected components in the complement of the extended separatrix skeleton are called *canonical regions of* X. Such a canonical region is found to be parallel, i.e. given either by a strip, an annular or spiral flow (see [4]). The union of the extended separatrix skeleton together with one orbit from each of the canonical regions is called the *completed separatrix skeleton.*

As a corollary of Theorem 1, Bendixson-Dulac Theorem, Poincaré-Bendixson Theorem and the hyperbolicity of the 2-saddle cycle, one can rule out limit cycles or prove their existence (see also Section 5), and hence the bifurcation diagram of global phase portraits can be completed.

THEOREM 5 ([6, 2]) *The bifurcation diagram of the 1-parameter family* $(X_m^1)_{m>0}$ *is as follows:*



$0 < m < m_C(1)$ *no limit cycles nor polycycles* | $m = m_C(1)$ *hyperbolic 2-saddle cycle*
$m_C(1) < m < 3/5$ *1 limit cycle, no polycycles* | $m \geq 3/5$ *no limit cycles nor polycycles*

4 Nilpotent Center-Focus Problem

A singularity is called a *center* if it has a punctured neighborhood full of non-isolated periodic orbits. The *Center-Focus Problem* aims at deciding whether a singularity is a center or a focus. Classically this problem deals with singularities being a center for the linearization of a polynomial or an analytic vector field (i.e. having purely imaginary eigenvalues), and is referred to as the Center Problem of Poincaré. A well-known classical result says that the analytic linear type center is proved to be a topological center if an analytic first integral exists.

Furthermore it is known that the Center-Focus Problem is algebraically solvable, also in the nilpotent case, by calculating (generalized) Lyapunov quantities (see [1]). However the expressions in the calculations often become too involved when the singularity changes stability, that in practice it is not at all an easy task to distinguish between a center or a focus.

Here, we decide between center and focus in a geometric-analytic way, relating it to the separatrix skeleton. Indeed, suppose that $m = m_S(k)$ and that the origin is a center for X_m^k. Then it follows that $m = m_C(k)$ and a stability analysis shows that for this parameter value the 2-saddle cycle is hyperbolic. This contradiction leads to the following result.

THEOREM 6 ([2]) *Let $k \geq 1$ and $m = m_S(k)$. The nilpotent singularity of $X^k_{m_S(k)}$ at the origin is a focus and not a center.*

The stability of the origin for $m = m_S(k)$ and the bifurcation of small amplitude limit cycles for $m \to m_S(k)$ for $k \geq 2$ is a work in progress in collaboration with Ilker Çolak.

5 Hilbert 16th Problem for $(X^k_m)_{m\in\mathbb{R}}$

Hilbert's 16th Problem asks, if it exists, for an upper bound for the number of limit cycles of a planar polynomial vector field $\dot{x} = P_n(x, y), \dot{y} = Q_n(x, y)$, only depending on the degree n of the polynomials P_n, Q_n. This problem is very vivid among specialists and a complete answer to it is not yet known. Dulac's problem, which concerns the finiteness of the number of limit cycles for *individual* analytic vector fields, is solved independently by Ilyashenko and Ecalle. Next result deals with the finiteness part of Hilbert's 16th Problem for $(X^k_m)_{m\in\mathbb{R}}$; i.e. the existence of a uniform upper bound for the number of limit cycles of $X^k_m, m \in \mathbb{R}$, only depending on k.

THEOREM 7 ([2]) *For all $k \geq 1$ there exists a finite number $\mathcal{H}(k)$ that bounds the number of limit cycles of X^k_m for all $m \in \mathbb{R}$. Any such upper bound, $\mathcal{H}(k)$, is at least one.*

In the rest of this section we sketch the proof of Theorem 7.

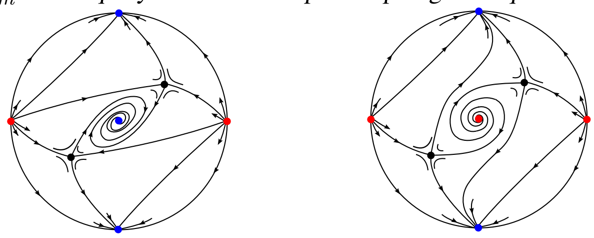
5.1 Roussarie compactification-localization method

In this section we indicate how Hilbert's 16th Problem for $(X^k_m)_{k\in\mathbb{R}}$ is reduced to so-called cyclicity problems, which are bifurcation problems of limit cycles, inside a compact family.

From [9] there exists the following equivalence between the global and local bounds for limit cycles, working with a compact analytic family of planar vector fields $(X_\lambda)_\lambda$: the number of limit cycles of X_λ in D is bounded uniformly with respect to $\lambda \in P$ if and only if for every limit periodic set of $(X_\lambda)_\lambda$ there are only finitely many limit cycles bifurcating from Γ.

By a *compact family of planar vector fields* $(X_\lambda)_\lambda$ we mean a family of vector fields that are defined on a compact metric space D, and that depend on a parameter λ, also belonging to a compact metric space P. A compact set Γ is called a limit periodic set of X_λ for $\lambda \to \lambda_0$ if and only if there exists a sequence $(\lambda_n)_{n\geq 1}$ with $\lambda_n \to \lambda_0$ for $n \to \infty$ such that for all $n \geq 1$ there exists a limit cycle γ_n of X_{λ_n} with $\gamma_n \to \Gamma$ when $n \to \infty$ (in the Hausdorff sense). There exists an analogue of the Poincaré-Bendixson Theorem determining the structure of limit periodic sets, in case that the analytic family $(X_\lambda)_\lambda$ has only a finite number of singularities. In that case, a limit periodic set is either a singular point, a periodic orbit or a graphic of X_{λ_0}. A proof of this structure theorem can be found in [9].

THEOREM 8 ([2]) *Let $k \geq 1$. There exist $0 < m_0(k) < m_\infty(k) < \infty$ such that X^k_m does not have limit cycles nor polycycles for $m < m_0(k)$ nor for $m > m_\infty(k)$. Furthermore, in these cases the global phase portrait of X^k_m is uniquely determined up to topological equivalence:*



$0 < m < m_0(k)$ $\qquad$ $m > m_\infty(k)$

For $m \leq 0$ it is found by means of a convenient Lyapunov function that the origin is a global attractor for X^k_m and hence X^k_m does not have any limit cycles. For small $m > 0$, in Subsection 5.2, we apply a generalization of Bendixson-Dulac Theorem to rule out limit cycles. For large m, in Subsection 5.3, we use a Roussarie compactification-localization procedure to localize the problem of global absence of limit cycles.

Hence by this theorem the global finiteness problem for $(X^k_m), m \in \mathbb{R}$ is reduced to the one on a compact parameter interval, $m \in [m_0(k), m_\infty(k)]$. By a Poincaré compactification of X^k_m, in Subsection 5.4, the global finiteness question of limit cycles of X^k_m for $m_0(k) \leq m \leq m_\infty(k)$ can thus be approached by local finiteness problems of limit cycles.

5.2 Proof of Theorem 8 for $m < m_0(k)$

For each m small enough we apply an interesting generalization of the Bendixson-Dulac Theorem that we recall from [6].

THEOREM 9 ([6]) *Let $X : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field on an open subset $\mathcal{U} \subset \mathbb{R}^2$ such that the boundary of $\mathcal{U}$ is formed by a finite union of algebraic curves. Assume that V is a rational function and $\lambda \in \mathbb{R}, \lambda > 0$ such that $M = \langle X, \nabla V\rangle - \lambda V \operatorname{div} X$ does not change sign on $\mathcal{U}$ and it vanishes only along a finite union of points and curves that are not invariant by the flow of X. (1) If all connected components of $\mathcal{U} \setminus \{V = 0\}$ are simply connected, then X has neither limit cycles nor polycycles entirely in $\mathcal{U}$. (2) If all connected components of $\mathcal{U} \setminus \{V = 0\}$ are simply connected, except one, say $\tilde{\mathcal{U}}$, that is 1-connected, then X has at most one limit cycle or polycycle in $\mathcal{U}$, that cannot coexist. Furthermore, if a limit cycle γ exists, then it is hyperbolic and $\gamma \subset \tilde{\mathcal{U}}$; the stability of γ is given by the sign of VM on $\tilde{\mathcal{U}}$.*

Returning to $X_m^{k,R}$, define $V_m(x,y) = 2m^{1/(k+1)}x^2 + y^4$ and

$$M(x,y,m) = \langle X_m^{k,R}(x,y), \nabla V_m(x,y)\rangle - \frac{2}{2k+1}V_m(x,y)\mathrm{div}X_m^{k,R}(x,y).$$

For m small enough it is found that $M(x,y,m) \geq 0$. Besides, the origin is the only maximal invariant set contained in $M(x,y,m) = 0$. Then by Theorem 9(2) there exists at most one limit cycle or polycycle, and both cannot coexist. Then, by Poincaré-Bendixson Theorem and a stability analysis of the origin and the polycycle, limit cycles as well as polycycles are ruled out for m small enough.

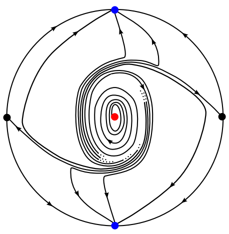
5.3 Proof of Theorem 8 for $m > m_\infty(k)$

The Roussarie localization method can evenly be used to obtain the global absence of limit cycles uniformly from local absence results (see [2]). For a limit periodic set Γ of (X_λ) for $\lambda \to \lambda_0$, we say that no limit cycles bifurcate from Γ if and only if there exists a neighborhood V_Γ of Γ in the Hausdorff sense and there exists a neighborhood $W_\Gamma \subset \mathbb{R}^p$ of λ_0 such that for all $\lambda \in W_\Gamma$ the vector field X_λ does not have any limit cycles in V_Γ.

For $m > 0$ the vector field X_m^k is topologically equivalent to

$$Y_\eta^{k,S} \leftrightarrow \dot{x} = y^3 - \eta x^{2k+1},\ \dot{y} = -x + y^{4k+1},\ \text{where } m\eta = 1.$$

Obviously, $(Y_\eta^{k,S})_{0<\eta\leq\eta_0}$ can analytically be extended to a compact analytic family $(\hat{Y}_\eta^{k,S})_{0\leq\eta\leq\eta_0}$ on the Poincaré disc. Using the Lyapunov function $V(x,y) = 2x^2 + y^4$ it is seen that the origin of $\hat{Y}_0^{k,S}$ exhibits a global repeller:



Therefore the only candidate limit periodic set of $\hat{Y}_\eta$ for $\eta \downarrow 0$ is the nilpotent focus at the origin. Then, clearly there are no large nor medium amplitude limit cycles for $\eta \downarrow 0$. This means that for every open ball $\mathcal{B}_0$ centered at the origin of $\mathbb{R}^2$, there exists $\eta_0 > 0$ such that for $0 \leq \eta \leq \eta_0$ there are no limit cycles of $\hat{Y}_\eta^{k,S}$ outside $\mathcal{B}_0$.

Considering the Poincaré map of first return using coordinates near the origin from a quasi-homogenous blow up, one shows that neither there are small amplitude limit cycles for $\eta \downarrow 0$. This means that there exists an open ball $\mathcal{B}_0$ centered at the origin of $\mathbb{R}^2$ and there exists $\eta_1 > 0$ such that for $0 \leq \eta < \eta_1$ there are no limit cycles of $\hat{Y}_\eta^{k,S}$ starting in $\mathcal{B}_0$.

5.4 Global finiteness for $X_m^k, m \in [m_0, m_\infty]$

As explained in Section 5.1 we reduce the global finiteness problem of limit cycles to local cyclicity problems. For a given $m^*(k) > 0$ limit periodic sets for the family $(\hat{X}_m^{k,R})_{m>0}$ for $m = m^*(k)$ can be $(0,0)$, a periodic orbit or a 2-saddle cycle (in the latter case $m^*(k) = m_C(k)$). The bifurcation problem of limit cycles from $(0,0)$ or a periodic orbit of $X_m^{k,R}$ is reduced to the bifurcation problem of fixed points of the analytic family of Poincaré first return maps. Therefore it is immediately seen that the number of limit cycles bifurcating from $(0,0)$ or a periodic orbit is finite. From [7] it follows that the number of limit cycles bifurcating from the hyperbolic 2-saddle cycle Γ inside $(X_m^{k,R})_{m>0}$ for m near $m_C(k)$ also is finite. It is to say, there exist an integer $N(k,\Gamma)$, positive constants m_1^k, m_2^k such that $m_C(k) \in (m_1^k, m_2^k)$ and a neighborhood $\mathcal{V}$ of Γ in the Hausdorff sense such that X_m^k has at most $N(k,\Gamma)$ limit cycles in $\mathcal{V}$ for all $m \in (m_1^k, m_2^k)$. Therefore all limit periodic sets generate at most a finite number of limit cycles in the family $(\hat{X}_m^{k,R})_{m_0^k \leq m \leq m_\infty^k}$. As a consequence, the Roussarie compactification-localization method guarantees the existence of a uniform upper bound $\mathcal{H}(k) < \infty$.

5.5 Lower bound for the Hilbert number

In fact, for $k = 1$, Theorem 7 follows from Theorem 5 with optimal upper bound $\mathcal{H}(1) = 1$. For general $k \geq 2$ it is seen that when m passes through $m_S(k)$ a Hopf-like bifurcation takes place. Then the focus at the origin changes its stability and at least one limit cycle is created for some values m near $m_S(k)$.

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