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Separatrix skeleton and limit cycles in some 1-parameter families of planar vector fields

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Abstract— We consider polynomial vector fields X_m^k of degree $4k+1$ given by $\dot{x} = y^3 - x^{2k+1}, \dot{y} = -x + my^{4k+1}$, $x, y \in \mathbb{R}$, where $m \in \mathbb{R}$ is a parameter. For any $k \geq 1$ we analyze the bifurcation of the separatrix skeleton of $(X_m^k)_{m>0}$ analytically. For $k = 1$ the bifurcation diagram of global phase portraits is deduced. Related to it, we address Hilbert's 16th Problem and the nilpotent Center-Focus Problem, restricting both problems to this 1-parameter family. This contribution summarizes the author's main results published in *Separatrix skeleton for some 1-parameter families of planar vector fields* in J Differ Equations **259** (2015). Additionally, we discuss the separatrix bifurcation for a generalization of the 1-parameter family $(X_m^k)_{m>0}$.

Keywords: planar vector field, separatrix skeleton, limit cycle, Hilbert's 16th problem, nilpotent center problem.

1 Introduction

We consider the 1-parameter family of planar vector fields

$$X_m^k \leftrightarrow \dot{x} = y^3 - x^{2k+1}, \dot{y} = -x + my^{4k+1}, \ (x, y) \in \mathbb{R}^2,$$

where $k \geq 1$ is an arbitrary but fixed integer and m a real parameter. During a conference on stability for differential equations held in Florence in 1985, Bacciotti asked for the stability type of the nilpotent singularity of X_m^k at the origin, and how its change of stability relates to the appearance of a polycycle. Shortly after that conference Galeotti and Gori presented a work in [5] on the stability type of the origin of X_m^k. More recently, Gasull, García and Giacomini reconsidered that problem using generalized Lyapunov focus quantities in [6]; furthermore they present a complete and rather technical study on limit cycles in the case that $k = 1$. In [2] the author completes the study of global phase portraits of X_m^1, by an analysis of the separatrix skeleton of X_m^k in function of m, for all $k \geq 1$ (see Sections 2 and 3). Besides, this study is used to exclude centers for X_m^k and to prove that the Hilbert number for $(X_m^k)_{m\in\mathbb{R}}$ is finite (see Sections 4 and 5). Both Hilbert's 16th Problem and the Center-Focus Problem are longstanding challenges from the beginning of the 20th century, and so-far a complete solution for them is not yet known beyond linear and quadratic differential equations respectively.

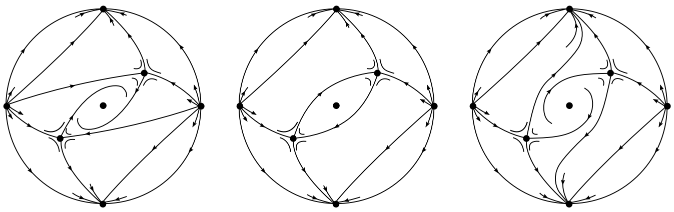
Here we recall the main results from [2] and we provide as well with outlines for their proofs, counting on a whole arsenal of local and global machinery from Qualitative Theory of Differential Equations. Additionally, we discuss some generalizations to replace X_m^k.

2 Separatrix skeleton for $k \geq 1$

2.1 Definitions and main result

Let X be a continuous planar vector field having only isolated singularities. An orbit Γ of X is called *separatrix* if it is homeomorphic to $\mathbb{R}$ and for each neighborhood $\mathcal{N}$ of Γ there exists $q \in \mathcal{N}$ such that $\alpha(q) \neq \alpha(\Gamma)$ or $\omega(q) \neq \omega(\Gamma)$. The closure of the union of separatrices is called the *separatrix skeleton of* X. In next theorem we present the key result from [2].

THEOREM 1 ([2]) *X_m^k undergoes a unique separatrix bifurcation for increasing $m > 0$, giving subsequently rise to the following three separatrix skeletons:*



In the subsequent subsections we prove the existence of a unique parameter value $m = m_C(k)$ at which X_m^k exhibits a 2-saddle cycle. For that aim we use a parameter dependent coordinate transformation that brings the family $(X_m^k)_{m>0}$ into a semi-complete family of indefinitely rotated vector fields.

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