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Algebraic curves of maximal cyclicity

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Abstract

The paper deals with analytic families of planar vector fields, studying methods to detect the cyclicity of a non-isolated closed orbit, i.e. the maximum number of limit cycles that can locally bifurcate from it. It is known that this multi-parameter problem can be reduced to a single-parameter one, in the sense that there exist analytic curves in parameter space along which the maximal cyclicity can be attained. In that case one speaks about a maximal cyclicity curve (mcc) in case only the number is considered and of a maximal multiplicity curve (mmc) in case the multiplicity is also taken into account. In view of obtaining efficient algorithms for detecting the cyclicity, we investigate whether such mcc or mmc can be algebraic or even linear depending on certain general properties of the families or of their associated Bautin ideal. In any case by well chosen examples we show that prudence is appropriate.

1. *Introduction*

1·1. *Motivation and description of the problem*

Hilbert's sixteenth problem essentially asks for the maximum number of limit cycles (isolated periodic orbits) in a polynomial vector field, depending uniformly on the degree. The finiteness part of Hilbert's 16th problem consists in proving that such a finite *global* bound exists. One way to handle this problem is to prove the existence of *local* upperbounds in *analytic* systems [**9**]. In this way, a global problem is transferred into local problems: the problem of bounding the maximum number of limit cycles γ, that can arise after small perturbations of X_{λ^0}, in small neighbourhoods of a limit periodic set Γ, inside a given analytic p-parameter family of planar vector fields $(X_\lambda)_\lambda, \lambda \sim \lambda^0$ [**9**]. This number is called the *cyclicity* of $(X_\lambda)_\lambda$ along Γ for $\lambda = \lambda^0$, and it is denoted by $Cycl(X_\lambda, (\Gamma, \lambda^0))$ or $Cycl$. Observe that in this paper, we essentially use the common notion of cyclicity, in which one does not count the multiplicity of the limit cycles. When the multiplicity is taken into account, we will refer to this number by *multiplicity* and it will shortly be denoted by $Mult(X_\lambda, (\Gamma, \lambda^0))$ or $Mult$. A more precise definition of this notion is given below in Section 1·2.

An effective tool in this approach is the Poincaré, or first-return mapping P_λ, and the associated displacement map $\delta_\lambda = P_\lambda - Id$. In this way, limit cycles of X_λ correspond to isolated zeroes of δ_λ. In this paper, as in [**2**], the limit periodic set Γ is supposed to be a regular non-isolated periodic orbit of X_{λ^0}. We say that X_{λ^0}