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Absolute cyclicity, Lyapunov quantities and center conditions

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ABSTRACT

In this paper we consider analytic vector fields X_0 having a non-degenerate center point e. We estimate the maximum number of small amplitude limit cycles, i.e., limit cycles that arise after small perturbations of X_0 from e. When the perturbation (X_λ) is fixed, this number is referred to as the cyclicity of X_λ at e for λ near 0. In this paper, we study the so-called absolute cyclicity; i.e., an upper bound for the cyclicity of any perturbation X_λ for which the set defined by the center conditions is a fixed linear variety. It is known that the zero-set of the Lyapunov quantities correspond to the center conditions (Caubergh and Dumortier (2004) [6]). If the ideal generated by the Lyapunov quantities is regular, then the absolute cyclicity is the dimension of this so-called Lyapunov ideal minus 1. Here we study the absolute cyclicity in case that the Lyapunov ideal is not regular.

1. Introduction

1.1. Cyclicity problem and center conditions

The existential part of Hilbert's 16th problem asks whether there exists a uniform upper bound for the number of limit cycles that appear in a planar polynomial vector field, only depending on its degree n. By the so-called Roussarie reduction this global problem is reduced to the investigation of local 'cyclicity problems'; in this reduction one looks for 'limit periodic sets', from which limit cycles can arise when slightly perturbing the vector field (cf. [15]). Let $(X_\lambda)_\lambda$ be an analytic family of vector fields, such that Γ is a limit periodic set of X_{λ^0}; then, the cyclicity of X_λ at (Γ, λ^0) is defined by

$$Cycl\big(X_\lambda, (\Gamma, \lambda^0)\big) = \lim_{\lambda \to \lambda^0} \sup_{\gamma \to \Gamma} \{\# \text{ limit cycles } \gamma \text{ of } X_\lambda\},$$

where the limit $\gamma \to \Gamma$ is taken in the sense of the Haussdorf distance. If for every given limit periodic set of an analytic family of vector fields, the cyclicity is finite, then there exists a uniform upper bound for the number of limit cycles of (X_λ).

There exist several (equivalent) techniques to study this number. Poincaré reduced the study of limit cycles to the study of zeroes of maps $(\delta_\lambda)_\lambda$, associated to the family of vector fields $(X_\lambda)_\lambda$ near the limit periodic set Γ. These maps are called displacement maps. In this paper we only consider analytic families of vector fields and isolated singularities; then, by Poincaré–Bendixson's theorem, a limit periodic set is one of the following compact invariant sets: a singularity, a periodic orbit or a graphic. The cyclicity in the first two cases corresponds to the local study of zeroes of an analytic family of maps; it is theoretically well understood. For instance, the cyclicity is finite; knowing a non-identically zero jet of finite order of the maps δ_{λ^0} at the limit periodic set, an explicit upper bound for the cyclicity is known and given in terms of the order

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