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GLOBAL CLASSIFICATION OF A CLASS OF CUBIC VECTOR FIELDS WHOSE CANONICAL REGIONS ARE PERIOD ANNULI

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We study cubic vector fields with inverse radial symmetry, i.e. of the form $\dot{x} = \delta x - y + ax^2 + bxy + cy^2 + \sigma(dx - y)(x^2 + y^2)$, $\dot{y} = x + \delta y + ex^2 + fxy + gy^2 + \sigma(x + dy)(x^2 + y^2)$, having a center at the origin and at infinity; we shortly call them cubic irs-systems. These systems are known to be Hamiltonian or reversible. Here we provide an improvement of the algorithm that characterizes these systems and we give a new normal form.

Our main result is the systematic classification of the global phase portraits of the cubic Hamiltonian irs-systems respecting time (i.e. $\sigma = 1$) up to topological and diffeomorphic equivalence. In particular, there are 22 (resp. 14) topologically different global phase portraits for the Hamiltonian (resp. reversible Hamiltonian) irs-systems on the Poincaré disc.

Finally we illustrate how to generalize our results to polynomial irs-systems of arbitrary degree. In particular, we study the bifurcation diagram of a 1-parameter subfamily of quintic Hamiltonian irs-systems. Moreover, we indicate how to construct a concrete reversible irs-system with a given configuration of singularities respecting their topological type and separatrix connections.

Keywords: Classification of global phase portraits; characterization of cubic centers; Lyapunov quantities; Hamiltonian planar vector fields; cubic vector fields.

1. Introduction

Let P and Q be two real polynomials in the variables x and y, then we say that $X = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a *planar polynomial vector field of degree* d if the maximum of the degrees of the polynomials P and Q is d. Such vector fields are called *quadratic* or *cubic* if $d = 2$ or $d = 3$, respectively. The *polynomial differential system* associated to the vector field X is

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y).$$

Two of the main classical problems in the qualitative theory of real planar polynomial vector fields are the determination of their limit cycles and the *center-focus problem*; i.e. to distinguish whether a singular point is either a focus or a center. A *center* is a singular point having a neighborhood fulfilled of periodic orbits with the unique exception of the singular point.

The classification of the centers of the polynomial vector fields is an old problem which started with the quadratic ones by the works of Dulac