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GLOBAL PHASE PORTRAITS OF SOME REVERSIBLE CUBIC CENTERS WITH COLLINEAR OR INFINITELY MANY SINGULARITIES

M. CAUBERGH*, J. LLIBRE† and J. TORREGROSA‡
Departament de Matemàtiques,
Edifici C. 08193 Bellaterra, Barcelona
**leen@mat.uab.cat*
†jllibre@mat.uab.cat
‡torre@mat.uab.cat

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We study the reversible cubic vector fields of the form $\dot{x} = -y + ax^2 + bxy + cy^2 - y(x^2 + y^2)$, $\dot{y} = x + dx^2 + exy + fy^2 + x(x^2 + y^2)$, having simultaneously a center at infinity and at the origin. In this paper, the subclass of these reversible systems having collinear or infinitely many singularities are classified with respect to topological equivalence.

Keywords: Center-focus problem; reversible planar vector fields; cubic vector fields; global classification of phase portraits.

1. Introduction

The center problem for polynomial differential equations in the plane is one of the celebrated and longstanding problems in the qualitative theory of planar differential equations (see e.g. the works [Dulac, 1908; Kapteyn, 1911, 1912; Bautin, 1954; Vulpe & Sibirskiĭ, 1988; Schlomiuk, 1993; Żołądek, 1994a, 1994b]). It asks to distinguish a linear center between a focus and a center of a polynomial vector field of a given degree. It is solved completely only for quadratic vector fields; for cubic vector fields, it is an open problem. This problem is closely related to another celebrated problem from the beginning of the twentieth century that is still one of the challenges for the twenty first century known as Hilbert's sixteenth problem, which asks essentially for the maximum number $H(n)$ of limit cycles of a polynomial vector field only depending on its degree n (see e.g. [Hilbert, 1902; Smale, 1998]). Until now, even $H(2)$ is not known. Limit cycles, i.e. isolated periodic orbits, can be found, for instance, by perturbing a center. The mechanism of simultaneous perturbation of different centers is used to construct concrete polynomial vector fields having a certain configuration of limit cycles. As such, lower bounds for the so-called Hilbert number $H(n)$ are deduced. For instance Shi [1980] used this mechanism to find $H(2) \geq 4$; in particular, the fourth limit cycle for a concrete quadratic vector field of an unbounded period annulus (center at infinity) was involved in the simultaneous bifurcation of quadratic centers.

In [Blows & Rousseau, 1993; Caubergh *et al.*, 2011] the problem of characterizing a center at the origin and simultaneously at infinity is solved for the following subfamily of the cubic differential equations

$$\begin{aligned} \dot{x} &= -y + ax^2 + bxy + cy^2 - y(x^2 + y^2), \\ \dot{y} &= x + dx^2 + exy + fy^2 + x(x^2 + y^2). \end{aligned} \tag{1}$$

By definition system (1) has a *center at infinity* if this system after transformation $x = \cos\theta/r$, $y = \sin\theta/r$ has a center at the origin. It turns out that vector fields of the family (1) that have