Relations between Abelian integrals and limit cycles

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Abstract

Limit cycles bifurcating in an unfolding from a *regular Hamiltonian cycle*, are in general directly controlled by the zeroes of the associated Abelian integral. Our purpose here is to investigate to what extent the Abelian integral allows one to study the limit cycles which bifurcate from a *singular Hamiltonian cycle*. We focus on the study of the 2-saddle cycles unfoldings. We show that the number of bifurcating limit cycles can exceed the number of zeroes of the related Abelian integral, even in generic unfoldings. However, in the case where one connection remains unbroken in the unfolding, we show how a finite codimension of the Abelian integral leads to a finite upper bound on the local cyclicity.

1 The problem

We consider C^∞ unfoldings of planar vector fields $(X_\lambda)_\lambda$, where $\lambda = (\bar{\lambda}, \varepsilon)$, $\bar{\lambda} \in (\mathbb{R}^k, 0)$, $\varepsilon \in [0, \varepsilon_0[$. We suppose that $X_{(\bar{\lambda},0)}$ is a Hamiltonian vector field X_H. So, there is a C^∞ function H such that

$$X_{(\bar{\lambda},0)} = X_H = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y}. \tag{1.1}$$

Hence we can write

$$X_\lambda = X_H + \varepsilon Y_{\bar{\lambda}} + o(\varepsilon), \quad \varepsilon \to 0. \tag{1.2}$$

We shall also consider the associated unfolding $(\omega_\lambda)_\lambda$ of dual 1-forms which can be written as

$$\omega_\lambda = dH + \varepsilon \nu_{\bar{\lambda}} + o(\varepsilon), \quad \varepsilon \to 0.$$

We suppose that X_H is a Hamiltonian vector field of *center type*. This means that there exists an annulus A of periodic orbits contained in $H^{-1}(]0, h_1[)$. More precisely, we assume

Y. Ilyashenko and C. Rousseau (eds.),
Normal Forms, Bifurcations and Finiteness Problems in Differential Equations, 1–32