Some criteria on the Uniqueness of Limit Cycles in planar polynomial vector fields

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In the qualitative theory of planar differential equations, research on limit cycles is an interesting and difficult topic. Limit cycles of planar vector fields were defined in the famous paper *Mémoire sur les courbes définies par une equation différentielle* of Poincaré [6]. At the end of the twenties van der Pol [7], Liénard [5] and Andronov [1] proved that a closed trajectory of a self-sustained oscillation occuring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After this observation, the nonexistence, existence, uniqueness and other properties of limit cycles have been studied extensively.

The more well-known method for proving the nonexistence of limit cycles in a simply connected region is the Bendixson-Dulac method (see Theorems 1.10 and 1.11 of [8]) and some variations of it, like Theorem 1.13 of [8]. The method of Dulac functions also gives upper bounds for the number of closed trajectories in a multiply connected region (see Theorem 1.12 of [8]). The Poincaré-Bendixson theorem states that a nonempty compact ω- or α-limit set of a planar flow, which contains no singular points, is a closed trajectory (see for instance [4]) this result allows to prove the annular region theorem which shows the existence of a limit cycle under convenient hypotheses.

The problem of uniqueness of a limit cycle for a given system is in general more difficult that the problem of existence. There are methods for uniqueness developped by Poincaré, Andronov, Cherkas, Levinson, Leontovich, Liénard, Massera, Sansone, Zhang, Zhifen and many others (see [8]). But in general these sufficient conditions of the previous methods are very restrictive. One of the best methods for studying the nonexistence, existence and uniqueness of limit cycles is analizing the Poincaré return map defined in a transversal section to the planar flow. But in general such analysis is not easy.

More difficult problems on the limit cycles appear when a planar system has more than one limit cycle, and when we try to understand their distribution on the plane. In fact the most famous problem on limit cycles is due to Hilbert [3]: *What is the maximum number of limit cycles for a polynomial vector field of degree n, and what are their relative positions?*.