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ON A RESULT OF DARBOUX

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Abstract

This paper is concerned with a relation of Darboux in enumerative geometry, which has very useful applications in the study of polynomial vector fields. The original statement of Darboux was not correct. The present paper gives two different elementary proofs of this relation. The first one follows the ideas of Darboux, and uses basic facts about the intersection index of two plane algebraic curves; the second proof is rather more sophisticated, and gives a stronger result, which should also be very useful. The power of the relation of Darboux is then illustrated by the provision of new, simple proofs of two known results. First, it is shown that an irreducible invariant algebraic curve of degree $n > 1$ without multiple points for a polynomial vector field of degree m satisfies $n \leqslant m + 1$. Second, a proof is given that quadratic polynomial vector fields have no algebraic limit cycles of degree 3.

1. *Introduction*

Darboux was the first to give the following relation in enumerative geometry [**2**, pp. 83–84]:

> On peut rattacher cette recherche à un lemme relatif à six polynômes A, A', B, B', C, C', de degrés l, l', m, m', n, n' satisfaisant à l'identité déjà considérée
> (48) $$AA' + BB' + CC' = 0;$$
> il est évident que les degrés des produits AA', BB', CC' sont égaux.
> On a donc déjà
> $$l + l' = m + m' = n + n' = \lambda.$$
> Cela posé, je dis que *la somme du nombre des points communs aux trois courbes*
> $$A = 0, \qquad B = 0, \qquad C = 0,$$
> *et du nombre des points communs aux trois courbes*
> $$A' = 0, \qquad B' = 0, \qquad C' = 0,$$
> *est égale à*
> $$\frac{lmn + l'm'n'}{\lambda}.$$

We shall refer to this result as the *Darboux lemma*; it can be stated more precisely as follows.

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