On the quartic algebraic solutions of quadratic systems

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Abstract

In this paper we give some properties and conditions for algebraic curves to be particular solutions of a polynomial differential system. In particular, we prove that quadratic systems cannot have a quartic algebraic particular solution composed by two ovals being both limit cycles of the system.

Introduction

We consider the real polynomial differential system

$$\dot{x} = P(x,y) \equiv \sum_{i=0}^{m} P_i(x,y), \quad \dot{y} = Q(x,y) \equiv \sum_{i=0}^{m} Q_i(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are coprime real polynomials of degree m, and $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous polynomials of degree $i = 0, 1, ..., m$. If $m = 2$ then system (1) is called *quadratic*.

Let $f(x,y) = 0$ be a real algebraic solution of degree n for system (1), that is

$$\frac{\partial f}{\partial x} P(x,y) + \frac{\partial f}{\partial y} Q(x,y) = k(x,y) f(x,y), \tag{2}$$

where $k(x,y) = \sum_{i=0}^{n} k_i(x,y)$ is a polynomial of degree at most $m-1$ called *cofactor* of $f = 0$ and $k_i(x,y)$ are homogeneous polynomials of degree $i = 0, 1, ..., m-1$. If the cofactor is identically zero then $f(x,y)$ is a polynomial first integral for system (1).

We say that (x_0, y_0) is a *singular point* of system (1) if it satisfies that $P(x_0,y_0) = Q(x_0,y_0) = 0$. A *limit cycle* of system (1) is an isolated periodic solution in the set of all periodic solutions of system (1).

Our main objective is to prove the following result:

Theorem 1 *A quartic algebraic curve with real coefficients being an algebraic solution of a quadratic system, cannot have two real ovals that are limit cycles of the system.*

Preliminary Results

We consider the projective coordinates $x = X/Z$, $y = Y/Z$. Let $f(x,y) = 0$ be an algebraic curve of degree n in the affine complex plane. This equation, in projective