Z_2-EQUIVARIANT LINEAR TYPE BI-CENTER CUBIC POLYNOMIAL HAMILTONIAN VECTOR FIELDS

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ABSTRACT. We study the global dynamical behavior of Z_2-equivariant cubic Hamiltonian vector fields with a linear type bi-center at $(\pm 1, 0)$. By using a series of symbolic computation tools, we obtain all possible phase portraits of these Z_2-equivariant Hamiltonian systems.

1. Introduction and statement of the main results

In the qualitative theory of planar vector fields the analysis when an equilibrium point p is either a center or a focus is one of the classical problems. We called it the center problem or the center-focus problem. Poincaré [30] and Dulac [14] defined that an equilibrium point p of a vector field in $\mathbb{R}^2$ is a *center* if it has a neighborhood U filled with periodic orbits with the unique exception of this equilibrium point. And a center p is *global* if $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits.

For polynomial vector fields the equilibrium point p in $\mathbb{R}^2$ is called *elementary* if at least one of the eigenvalues of the Jacobian matrix (or the linear part) at this point is nonzero, otherwise it is called *non-elementary*. We say that a non-elementary equilibrium point is *nilpotent* if its two eigenvalues are zero but its Jacobian matrix is not identically zero, and *degenerate* when the Jacobian matrix is identically zero. Correspondingly, if a polynomial vector field in $\mathbb{R}^2$ has a center at the origin, after making a time rescaling and a linear change of variables, this system can be written as

$$\dot{x} = \sum_{k=1}^{n} X_k(x,y), \qquad \dot{y} = \sum_{k=1}^{n} Y_k(x,y), \tag{1}$$

where $\dot{\{\}} := \mathrm{d}/\mathrm{d}t$, t is the time,

$$(X_1(x,y), Y_1(x,y)) = \begin{cases} (0,\ 0)\,, \\ (y,\ 0)\,, \\ (-y,\ x)\,, \end{cases} \tag{2}$$

$X_k(x,y)$ and $Y_k(x,y)$ are real homogeneous polynomials of degree $k\,(k \geq 2)$ in x and y. Then the center at the origin is called a *degenerate center*, a *nilpotent center* and a *linear type center* when (X_1, Y_1) satisfies the right hand side of (2) respectively. Some algorithms for the characterization of these three type centers have been studied in [5, 11, 17, 26, 31].

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As we know, a great deal of work has been done for the classifications of centers for the vector fields in $\mathbb{R}^2$ during more than a century. The center problem of quadratic polynomial systems started with the results of Bautin [4], Kapteyn [18, 19] and Zoladek [38]. The classification of centers for some cubic polynomial vector fields were studied in [1, 8, 27, 34] and references therein. But it is very difficult to provide the complete characterization of the centers for all cubic polynomial vector fields, and it is unsolved problem. There has been partial results about the centers of the quartic and the quintic polynomial vector fields (see [6, 7]). Recently, the phase portraits of some cubic Hamiltonian vector fields with a nilpotent center or a linear type center at the origin were classified in [12, 13, 16].

However there are very few families of the vector fields about the existence of multiple centers. In this work we study a class of planar Z_q-equivariant polynomial vector fields, whose phase portraits are unchanged by a rotation of $2\pi/q$ radians around one point, where q is a positive integer. The study of Z_q-equivariant polynomial systems is very importance for the Hilbert's 16th problem, for more details see [20, 21, 22]. Yu and Han [36, 37], Liu and Huang [24] studied the limit cycle bifurcations for several classes of Z_2-equivariant cubic polynomial systems. On the other hand we say that Z_q-equivariant polynomial systems have a bi-center at the equilibrium points p_1 and p_2 if these two equilibrium points are both centers. We will call it the bi-center problem. The authors of [23, 32] studied the bi-center problem for some Z_2-equivariant polynomial systems. Chen et al. [9] investigated the local critical bifurcation in a class of Z_2-equivariant cubic polynomial systems with a bi-center. But they did not provide the corresponding phase portraits of these vector fields.

To characterize the phase portraits of a planar polynomial vector field in the Poincaré disc, we need to know the local phase portraits of the infinite and finite equilibria of these systems using the Poincaré compactification. And we will introduce a series of methods to characterize the equilibria of the polynomial systems. Then we apply these methods to study the global dynamics of a class of Z_2-equivariant cubic Hamiltonian vector fields with a linear type bi-center at $(\pm 1, 0)$, described by

$$\begin{aligned} \dot{x} &= -(1+a_{21})y + a_{21}x^2y - 3b_{03}xy^2 + a_{03}y^3, \\ \dot{y} &= -x + x^3 - a_{21}xy^2 + b_{03}y^3. \end{aligned} \tag{3}$$

In Section 3 we will show how to obtain the Z_2-equivariant cubic Hamiltonian systems (3).

For determining the global flow of these polynomial vector fields we need to characterize their separatrices. The *separatrices* in the Poincaré disc $\mathbb{D}^2$ include all the finite equilibria, all the infinite orbits, all the limit cycles, and all the separatrices of the hyperbolic sectors of the infinite and finite equilibria. If we denote by Σ the closed set formed by all the separatrices of the vector field, the components of $\mathbb{D}^2 \setminus \Sigma$ are called the *canonical regions*. The *separatrix squeleton* of a polynomial vector field is formed by Σ union one orbit for each canonical region. The phase portraits of two polynomial vector fields in the Poincaré disc are topological equivalent if and only if the two corresponding separatrix squeletons are topological equivalent, see for more details [29]. We denote by s the number of separatrices, by r the number of canonical regions. For more details about the Poincaré compactification see [10, 15]. And in the phase portraits of the following theorem a line of the form "$\cdots\cdots$" denotes that it is filled of equilibria of a Hamiltonian system (3). We state the main result of this paper.

Theorem 1.1. *The phase portraits in the Poincaré disc of the Z_2-equivariant cubic Hamiltonian systems* (3) *with a linear type bi-center at* $(\pm 1, 0)$ *are topologically equivalent to the 36 phase portraits described in Figures* 1 *and* 2.

In the next section we will give some preliminaries about the Poincaré compactification for analyzing the equilibrium points. In Section 3 we provide how to obtain the Z_2-equivariant cubic Hamiltonian systems (3). In Section 4 we characterize the global phase portraits of systems (3) in the Poincaré disc, that is we prove Theorem 1.1.

2. Preliminaries

In this section we introduce some preliminaries about the Poincaré compactification and the equilibria. These are described in Chapter 5 of [15]. This compactification is very helpful for studying the phase portraits of the planar polynomial vector fields.

We denote by $\mathbb{R}^2$ the plane in $\mathbb{R}^3$ defined with the points $\mathbf{s} = (s_1, s_2, s_3) = (x_1, x_2, 1)$. Let $P(\mathbb{R}^2)$ be the set of the polynomial vector fields $\mathcal{X}$ in $\mathbb{R}^2$ of the form

$$(\dot{x_1}, \dot{x_2}) = (X(x_1, x_2), Y(x_1, x_2)). \tag{4}$$

The degree d of $\mathcal{X}$ equals $\max\{\deg X, \deg Y\}$. Let $\mathbb{S}^2 = \{\mathbf{s} \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$ be the *Poincaré sphere*. The plane $\mathbb{R}^2$ is tangent to $\mathbb{S}^2$ at the point $(0, 0, 1)$. We denote by $\mathbb{S}^1 = \{\mathbf{s} \in \mathbb{S}^2 : \ s_3 = 0\}$ the equator of $\mathbb{S}^2$ which corresponds to the infinity of $\mathbb{R}^2$. For the Poincaré compactification $p(\mathcal{X})$ of $\mathcal{X}$ one close disc $\mathbb{D}^2 = \{\mathbf{s} \in \mathbb{R}^2 : s_1^2 + s_2^2 \leq 1\}$ is called the *Poincaré disc*. The equilibria in $\mathbb{D}^2$ lying on $\mathbb{S}^1$, i.e. the boundary of the disc $\mathbb{D}^2$, are the corresponding *infinite equilibria* of $\mathcal{X}$. The equilibria in the interior of the Poincaré disc, i.e. on $\mathbb{D}^2 \setminus \mathbb{S}^1$, are the corresponding *finite equilibria* of $\mathcal{X}$. Then we provide the expressions of the Poincaré compactification for the polynomial differential systems.

In order to draw the phase portraits of the infinite equilibria of $\mathbb{S}^1$ we only need to consider the local charts $U_i = \{\mathbf{s} \in \mathbb{D}^2 : \ s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{D}^2 : \ s_i < 0\}$, for $i = 1, 2$, with the corresponding diffeomorphisms

$$\varphi_i : \ U_i \to \mathbb{R}^2, \qquad \psi_i : \ V_i \to \mathbb{R}^2, \tag{5}$$

defined by

$$\varphi_i(\mathbf{s}) = -\psi_i(\mathbf{s}) = (\frac{s_m}{s_i}, \frac{s_n}{s_i}) = (u, v)$$

for $m < n$ and $m, n \neq i$. Thus (u, v) will play different roles in these two local charts. The expression of $p(\mathcal{X})$ in U_1 is

$$(\dot{u}, \dot{v}) = (v^d(Y_1 - uX_1), -v^{d+1}X_1), \tag{6}$$

where $X_1 = X(1/v, u/v)$ and $Y_1 = Y(1/v, u/v)$. In U_2 it is

$$(\dot{u}, \dot{v}) = (v^d(X_2 - uY_2), -v^{d+1}Y_2), \tag{7}$$

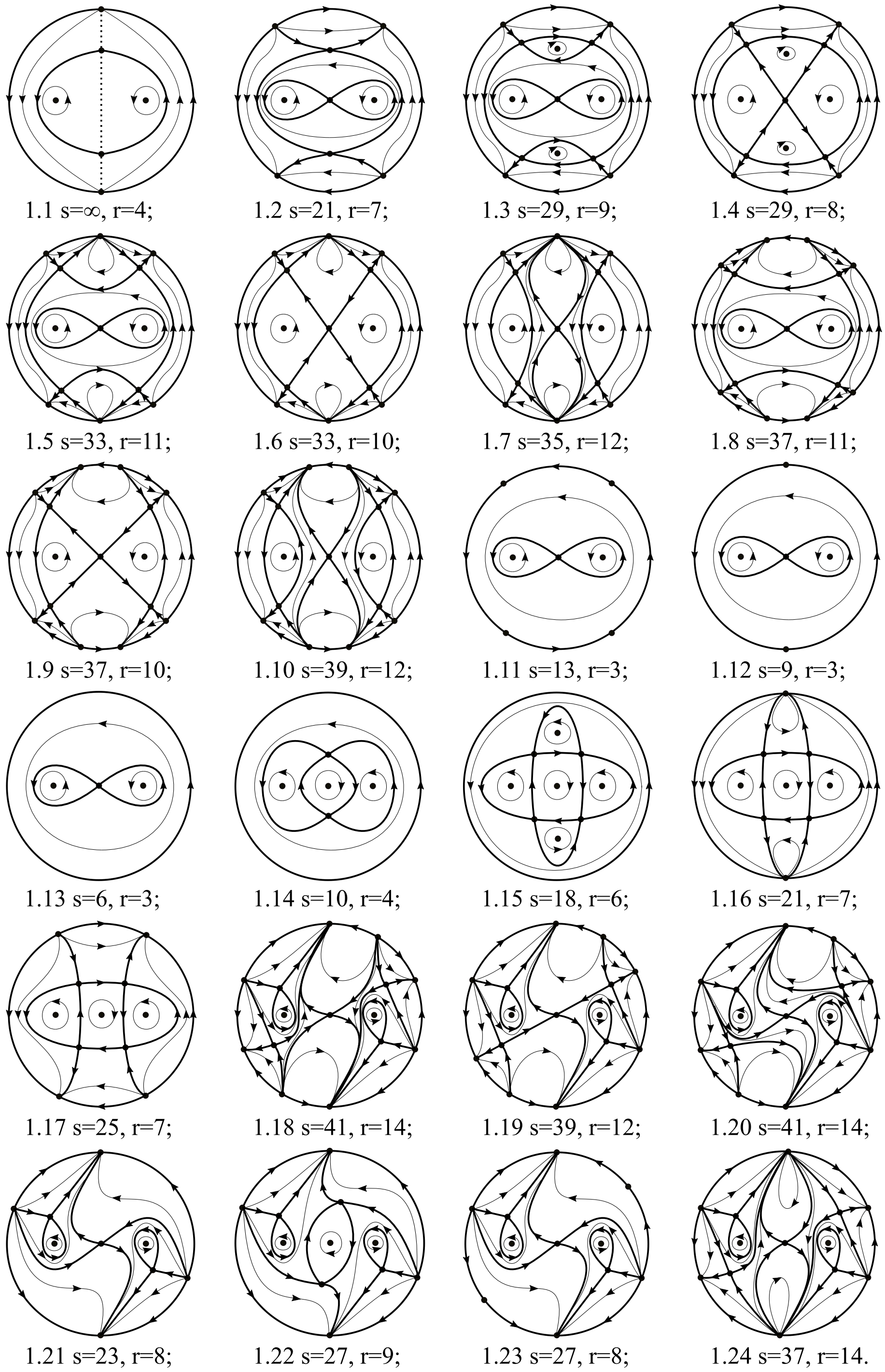
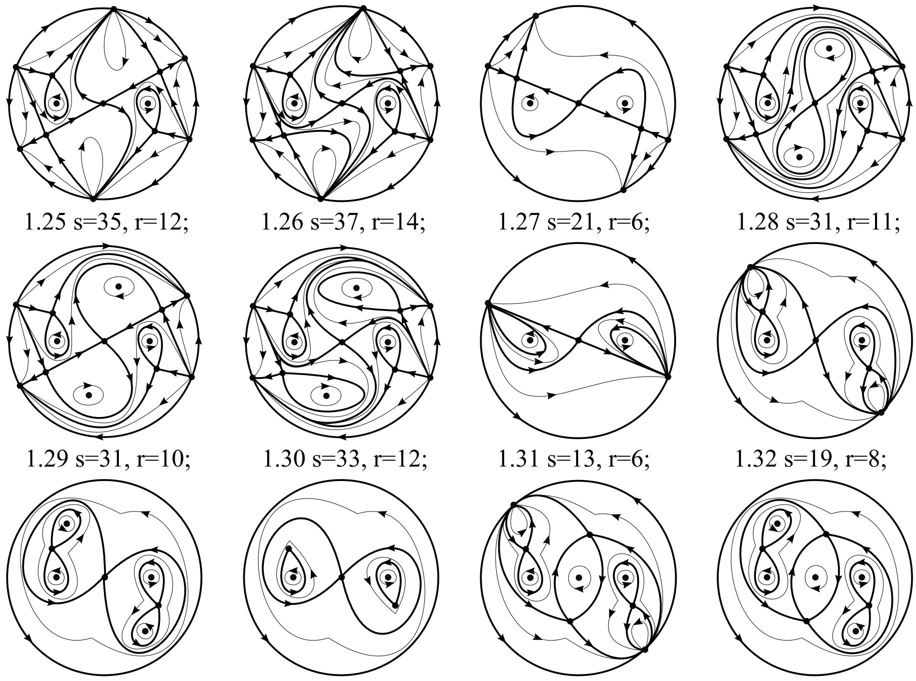


FIGURE 1. The topological phase portraits 1.1-1.24 in the Poincaré disc of Theorem 1.1.



1.25 s=35, r=12; 1.26 s=37, r=14; 1.27 s=21, r=6; 1.28 s=31, r=11;

1.29 s=31, r=10; 1.30 s=33, r=12; 1.31 s=13, r=6; 1.32 s=19, r=8;

1.33 s=14, r=7; 1.34 s=10, r=5; 1.35 s=23, r=9; 1.36 s=18, r=8.

FIGURE 2. The topological phase portraits 1.25-1.36 in the Poincaré disc of Theorem 1.1.

where $X_2 = X(u/v, 1/v)$ and $Y_2 = Y(u/v, 1/v)$. It is sufficient to study the equilibria at $U_1|_{v=0}$ and at the origin of U_2 for studying all the infinite equilibria of $\mathcal{X}$. The expressions of $p(\mathcal{X})$ in V_i, $i = 1, 2$ are those in U_i multiplied by $(-1)^{d-1}$. Note that if $\mathbf{s} \in \mathbb{S}^1$ is an infinite equilibrium point, $-\mathbf{s} \in \mathbb{S}^1$ is also an infinite equilibrium point. Hence the number of infinite equilibria is even.

Next we introduce the topological index of an equilibrium point, which is one useful tool to determine the type of the equilibria. Here we will present two important theorems, the Index Poincaré Formula and the Poincaré–Hopf Theorem, for more details see Chapter 6 of [15].

Theorem 2.1. *We denote by p an isolated equilibrium point with the finite sectorial decomposition property. Let q, h and e be the number of parabolic, hyperbolic and elliptic sectors of p, respectively. Then the topological index of the equilibrium point p equals* $1 + (e - h)/2$.

Corollary 2.2. *The topological indices of a center, a cusp, a saddle and a node equal* 1, 0, -1 *and* 1, *respectively.*

Theorem 2.3. *For any continuous vector field on the sphere* $\mathbb{S}^2$ *with finitely many equilibria, the sum of their topological indices is* 2.

3. Obtaining systems (3)

Without loss of generality the Z_2-equivariant cubic Hamiltonian systems here considered are obtained from the differential systems

$$\begin{aligned}\dot{x} =&a_{10}x + a_{01}y + a_{03}y^3 + a_{12}xy^2 + a_{21}x^2y + a_{30}x^3 = X(x, y),\\ \dot{y} =&b_{10}x + b_{01}y + b_{03}y^3 + b_{12}xy^2 + b_{21}x^2y + b_{30}x^3 = Y(x, y).\end{aligned} \tag{8}$$

Suppose $(\pm 1, 0)$ are two equilibrium points of systems (8). Then we have

$$a_{10} = -a_{30}, \qquad b_{10} = -b_{30}, \tag{9}$$

and the Jacobian matrix of systems (8) at $(\pm 1, 0)$ is given by

$$J = \begin{pmatrix} 2a_{30} & a_{01} + a_{21} \\ 2b_{30} & b_{01} + b_{21} \end{pmatrix}. \tag{10}$$

Proposition 3.1. *Assume $a_{10} = -a_{30}$ and $b_{10} = -b_{30}$. The equilibrium points $(\pm 1, 0)$ are two linear type centers of Hamiltonian systems (8) if and only if*

$$a_{12} + 3b_{03} = a_{21} + b_{12} = 3a_{30} + b_{21} = b_{01} - a_{30} = 0,$$

and $2a_{30}^2 + (a_{01} + a_{21})b_{30} < 0$.

Proof. If the equilibrium points $(\pm 1, 0)$ are two linear type centers of systems (8), we have the trace and the determinant of (10) at $(\pm 1, 0)$ are zero and positive, respectively. Thus, we obtain $b_{01} = -b_{21} - 2a_{30}$ and $2a_{30}^2 + (a_{01} + a_{21})b_{30} < 0$.

Next, let $H(x, y)$ be the quartic Hamiltonian of systems (8). To find this Hamiltonian, we integrate $X(x, y)$ of (8) with respect to y and obtain

$$\begin{aligned}H_1(x, y) =&f(x) + \int X(x, y)dy\\ =&f(x) - a_{30}xy + a_{30}x^3y + \frac{a_{01}}{2}y^2 + \frac{a_{21}}{2}x^2y^2 + \frac{a_{12}}{3}xy^3 + \frac{a_{03}}{4}y^4,\end{aligned} \tag{11}$$

for some real polynomials f. And we integrate $Y(x, y)$ of (8) with respect to x and obtain

$$\begin{aligned}H_2(x, y) =&g(y) - \int Y(x, y)dx\\ =&g(y) + \frac{b_{30}}{2}x^2 - \frac{b_{30}}{4}x^4 + (2a_{30} + b_{21})xy - \frac{b_{21}}{3}x^3y\\ &- \frac{b_{12}}{2}x^2y^2 - b_{03}xy^3,\end{aligned} \tag{12}$$

for some real polynomials g. Equating $H_1(x, y)$ to $H_2(x, y)$ we obtain

$$a_{12} + 3b_{03} = a_{21} + b_{12} = 3a_{30} + b_{21} = 0, \tag{13}$$

$f(x) = b_{30}x^2/2 - b_{30}x^4/4$ and $g(y) = a_{01}y^2/2 + a_{03}y^4/4$. Therefore the statement of this proposition holds if and only if (13), $b_{01} = a_{30}$ and $2a_{30}^2 + (a_{01} + a_{21})b_{30} < 0$. □

Then we take $a_{30} = 0$, $a_{01} + a_{21} = -1$ and $b_{30} = 1$, so that systems (8) become the Z_2-equivariant cubic Hamiltonian systems (3) having the Hamiltonian

$$H(x, y) = \frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1 + a_{21}}{2}y^2 + \frac{a_{03}}{4}y^4 + \frac{a_{21}}{2}x^2y^2 - b_{03}xy^3. \tag{14}$$

Doing the transformation

$$x = \frac{\sqrt{2}x_1}{2} + 1, \qquad y = y_1, \qquad t = \sqrt{2}\tau \tag{15}$$

systems (3) can be written as

$$\begin{aligned} x_1' =& - y_1 + \sqrt{2}a_{21}x_1y_1 + \frac{a_{21}}{2}x_1^2y_1 - 3b_{03}y_1^2 - \frac{3\sqrt{2}b_{03}}{2}x_1y_1^2 + a_{03}y_1^3, \\ y_1' =& x_1 + \frac{3\sqrt{2}}{4}x_1^2 + \frac{1}{4}x_1^3 - \frac{\sqrt{2}a_{21}}{2}y_1^2 - \frac{a_{21}}{2}x_1y_1^2 + \frac{\sqrt{2}b_{03}}{2}y_1^3, \end{aligned} \tag{16}$$

where $\{'\} := \mathrm{d}/\mathrm{d}\tau$. Note that the equilibrium point $(1, 0)$ of systems (3) corresponds to the origin of systems (16), which is a center-focus type point.

Since the flow of Hamiltonian systems preserves the area, any finite equilibrium point of Hamiltonian systems must be either a center, or a union of an even number of hyperbolic sectors. In particular, any finite nilpotent equilibrium point of Hamiltonian systems is either a cusp, a center or a saddle, for more details see Theorem 3.5 of [15]. Then the equilibrium points $(\pm 1, 0)$ of systems (3) cannot be foci, so that they are two linear type centers. In addition we do not need to determine the limit cycles of systems (3) because the existence of Hamiltonian prevents the existence of limit cycles.

4. Global phase portraits of systems (3)

Note that we can assume $b_{03} \geq 0$ because the Z_2-equivariant cubic Hamiltonian systems (3) are invariant under the following transformation

$$(x, y, t, a_{21}, a_{03}, b_{03}) \to (-x, y, -t, a_{21}, a_{03}, -b_{03}).$$

Thus we study the phase portraits of systems (3) in two different cases $b_{03} = 0$ and $b_{03} > 0$.

4.1. Phase portraits when $b_{03} = 0$.

Systems (3) are

$$\begin{aligned} \dot{x} =& - (1 + a_{21})y + a_{21}x^2y + a_{03}y^3, \\ \dot{y} =& - x + x^3 - a_{21}xy^2. \end{aligned} \tag{17}$$

It is easy to see that systems (17) are symmetric with respect to the x-axis and the y-axis. We calculate the resultant of $\dot{x}$ and $\dot{y}$ with respect to x, and obtain

$$\mathrm{Res}[\dot{x}, \dot{y}, x] = y^3(-1 - a_{21} + a_{03}y^2)(-1 + a_{03}y^2 + a_{21}^2y^2)^2. \tag{18}$$

For (18) to be identically zero, it needs $a_{21} = -1$ and $a_{03} = 0$. Then systems (17) have non-isolated equilibrium points. Thus, we consider the following two subcases $a_{03}^2 + (a_{21} + 1)^2 = 0$ and $a_{03}^2 + (a_{21} + 1)^2 \neq 0$.

4.1.1. *Subcase* $a_{03}^2 + (a_{21} + 1)^2 = 0$.

Then systems (17) become

$$\begin{aligned} \dot{x} =& - x^2y, \\ \dot{y} =& - x + x^3 + xy^2. \end{aligned} \tag{19}$$

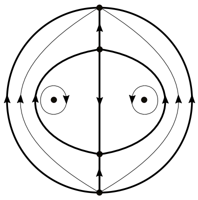


FIGURE 3. The phase portrait of system (20).

And the y-axis is filled of equilibrium points of system (19). By the rescaling time $\mathrm{d}t = x\mathrm{d}\tau$ system (19) is given by

$$x' = -xy, \qquad y' = -1 + x^2 + y^2, \tag{20}$$

which has one invariant straight line $x = 0$. Thus, we can analyze the phase portrait of (20) for studying the associated Hamiltonian system (19).

First we study the infinite equilibria of (20) using the Poincaré compactification. In the local chart U_1 system (20) becomes

$$u' = 1 + 2u^2 - v^2, \qquad v' = uv. \tag{21}$$

When $v = 0$ system (21) has no equilibrium points. In the local chart U_2 system (20) writes

$$u' = -u(2 + u^2 - v^2), \qquad v' = v(-1 - u^2 + v^2). \tag{22}$$

The origin of (22) is an equilibrium point, which is an attracting node. Hence system (20) has only two infinite equilibrium points, the origins of the local charts U_2 and V_2. Since the degree of system (20) is 2, the flow in a neighborhood of the origin of V_2 has the opposite sense with respect to the flow in a neighborhood of the origin of U_2. This completes the local phase portraits at the infinite equilibria of system (20).

Next we study the finite equilibria of system (20). System (20) has two equilibria $p_{3,4} = (0, \pm 1)$ different from the bi-center at $p_{1,2} = (\pm 1, 0)$. Since the eigenvalues of the Jacobian matrix of (20) at p_3 are $\lambda_1 = 2$ and $\lambda_2 = -1$ the equilibrium point p_3 is a saddle. Similarly we have that p_4 is also a saddle. Since the y-axis is an invariant straight line and system (19) has no more finite equilibrium points, the saddles $p_{3,4}$ must be on the boundary of the region formed by all the period annuluses surrounding the center p_1, this region is called the *period annulus* of the center p_1. In short, by the symmetries with respect the x-axis and the y-axis we obtain that the phase portrait of system (20) is the one shown in Figure 3. Going back to the Z_2-equivariant cubic Hamiltonian system (19), the y-axis is filled of equilibrium points of system (19), and due to the rescaling time $\mathrm{d}t = x\mathrm{d}\tau$ the orbits of system (19) has the opposite orientation with respect to the ones of system (20) in the region $x < 0$. Consequently, the global phase portrait of system (19) in Poincaré disc is topologically equivalent to the one 1.1 of Figure 1.

4.1.2. *Subcase* $a_{03}^2 + (a_{21}+1)^2 \neq 0$. Using (6) systems (17) in the local chart U_1 become

$$\begin{aligned} \dot{u} =& 1 - 2a_{21}u^2 - v^2 - a_{03}u^4 + (1+a_{21})u^2v^2, \\ \dot{v} =& uv(-a_{21} - a_{03}u^2 + v^2 + a_{21}v^2). \end{aligned} \tag{23}$$

Then the infinite equilibria $(u,0)$ of systems (23) in the local chart U_1 must satisfy $1 - 2a_{21}u^2 - a_{03}u^4 = 0$. Moreover the Jacobian matrix of systems (23) at $(u,0)$ is

$$M = \begin{pmatrix} -4u(a_{21}+a_{03}u^2) & 0 \\ 0 & -u(a_{21}+a_{03}u^2) \end{pmatrix}. \tag{24}$$

Proposition 4.1. *For systems* (23) *with* $a_{03}^2 + (a_{21}+1)^2 \neq 0$ *the following statements hold.*

(I) *If* $a_{03} < -a_{21}^2$ *and* $a_{21} > 0$, *or* $a_{03} \le 0$ *and* $a_{21} \le 0$ *systems* (23) *have no infinite equilibrium points.*
(II) *If* $a_{03} > 0$ *systems* (23) *have two infinite equilibrium points* $A_{1,2}$, *which are an attracting node and a repelling node, respectively.*
(III) *If* $a_{03} = 0$ *and* $a_{21} > 0$ *systems* (23) *have two infinite equilibrium points* $(\pm\sqrt{2a_{21}}/2a_{21}, 0)$, *which are an attracting node and a repelling node, respectively.*
(IV) *If* $0 > a_{03} > -a_{21}^2$ *and* $a_{21} > 0$ *systems* (23) *have four infinite equilibrium points* $A_{1,2,3,4}$, *where* $A_{1,4}$ *are two attracting nodes and* $A_{2,3}$ *are two repelling nodes.*
(V) *If* $a_{03} = -a_{21}^2$ *and* $a_{21} > 0$ *systems* (23) *have two infinite equilibrium points* $(\pm\sqrt{a_{21}}/a_{21}, 0)$, *whose local phase portraits both consist of two hyperbolic sectors, see Figure 4(c).*

Here we have

$$A_{1,2} = \left(\pm\sqrt{(-a_{21} + \sqrt{a_{03} + a_{21}^2})/a_{03}}, 0\right)$$

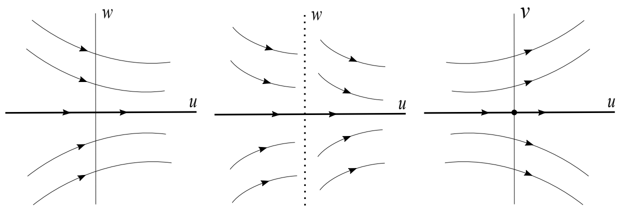
and

$$A_{3,4} = \left(\pm\sqrt{(-a_{21} - \sqrt{a_{03} + a_{21}^2})/a_{03}}, 0\right).$$

Proof. Statements (I) to (IV) are easy to prove. They just need to check the roots of $1 - 2a_{21}u^2 - a_{03}u^4 = 0$ and compute the traces and the determinants of (24) at these equilibrium points.

Now we consider case (V). If $a_{03} = -a_{21}^2$ and $a_{21} > 0$ the infinite equilibrium points $(\pm\sqrt{a_{21}}/a_{21}, 0)$ are degenerate. In order to study the local phase portraits of these two infinite equilibria we need to do some changes of variables called *blow-up's*, for a detailed introduction to the blow-up's changes see for instance [2]. Due to the symmetries of the systems we only need to analyze the local phase at the equilibrium point $(\sqrt{a_{21}}/a_{21}, 0)$. Doing the change of variables $(u,v) \to (u + \sqrt{a_{21}}/a_{21}, v)$ the equilibrium point $(\sqrt{a_{21}}/a_{21}, 0)$ is translated to the origin of coordinates, and we obtain the systems

$$\begin{aligned} \dot{u} =& 1 - 2a_{21}(u + \sqrt{a_{21}}/a_{21})^2 + a_{21}^2(u + \sqrt{a_{21}}/a_{21})^4 - v^2 \\ & + (u + \sqrt{a_{21}}/a_{21})^2v^2 + a_{21}(u + \sqrt{a_{21}}/a_{21})^2v^2, \\ \dot{v} =& v(u + \sqrt{a_{21}}/a_{21})(-a_{21} + a_{21}^2(u + \sqrt{a_{21}}/a_{21})^2 + v^2 + a_{21}v^2). \end{aligned} \tag{25}$$



(a) (b) (c)

FIGURE 4. Blow-up at the origin of (25). (a) Systems (27); (b) Systems (26); (c) Systems (25).

We do the blow-up $(u, v) \to (u, w)$ with $w = v/u$ for analyzing the local phase portrait at the origin of systems (25). And we obtain the systems

$$
\begin{aligned}
\dot{u} =& \frac{1}{a_{21}} u^2 (4a_{21}^2 + w^2 + 4a_{21}^2\sqrt{a_{21}}u + 2\sqrt{a_{21}}w^2u + 2a_{21}\sqrt{a_{21}}w^2u + a_{21}^3u^2 \\
& + a_{21}w^2u^2 + a_{21}^2w^2u^2), \\
\dot{w} =& -\frac{1}{a_{21}} wu^2 (2a_{21}^2 + w^2 + a_{21}^2\sqrt{a_{21}}u + \sqrt{a_{21}}w^2u + a_{21}\sqrt{a_{21}}w^2u).
\end{aligned} \tag{26}
$$

By the rescaling of the time $\mathrm{d}t = a_{21}\mathrm{d}\tau/u^2$ we have

$$
\begin{aligned}
u' =& 4a_{21}^2 + w^2 + 4a_{21}^2\sqrt{a_{21}}u + 2\sqrt{a_{21}}w^2u + 2a_{21}\sqrt{a_{21}}w^2u + a_{21}^3u^2 \\
& + a_{21}w^2u^2 + a_{21}^2w^2u^2, \\
w' =& -w(2a_{21}^2 + w^2 + a_{21}^2\sqrt{a_{21}}u + \sqrt{a_{21}}w^2u + a_{21}\sqrt{a_{21}}w^2u).
\end{aligned} \tag{27}
$$

Systems (27) have no equilibrium points for $u = 0$. Thus we study the local phase portrait in the neighborhood of $u = 0$. Note that $u'|_{w=0} = 4a_{21}^2 + O(u)$ showing that the flow is increasing in the direction u, and $w'|_{u=0} = -2a_{21}^2w + O(w^2)$ showing that the flow is decreasing and increasing in the direction w when $w > 0$ and $w < 0$ (see Figure 4(a)). Going back through the changes of variables until systems (25), as in the pass from systems (27) to systems (26) does not change the orientation of flows of systems (26) except that $u = 0$ is filled with equilibrium points, see Figure 4(b). And taking into account the behavior of the flows on the axes $\dot{u}|_{v=0} = a_{21}u^2(2 + \sqrt{a_{21}}u)^2 > 0$ and $\dot{v}|_{u=0} = \sqrt{a_{21}}(1 + a_{21})v^3/a_{21}$, we obtain that the phase portrait at the origin of systems (25) consists of two hyperbolic sectors (see Figure 4(c)). □

We proceed to study when the origin of the local chart U_2 is an infinite equilibrium. Systems (17) in the local chart U_2 acquire the form

$$
\dot{u} = a_{03} + 2a_{21}u^2 - u^4 - (1 + a_{21})v^2 + u^2v^2, \qquad \dot{v} = uv(a_{21} - u^2 + v^2). \tag{28}
$$

Proposition 4.2. *For systems* (28) *with* $a_{03}^2 + (a_{21} + 1)^2 \neq 0$ *the following statements hold.*

(I) *If* $a_{03} \neq 0$ *systems* (28) *have no infinite equilibrium points.*
(II) *If* $a_{03} = 0$ *systems* (28) *have an infinite equilibrium point at the origin. Furthermore:*

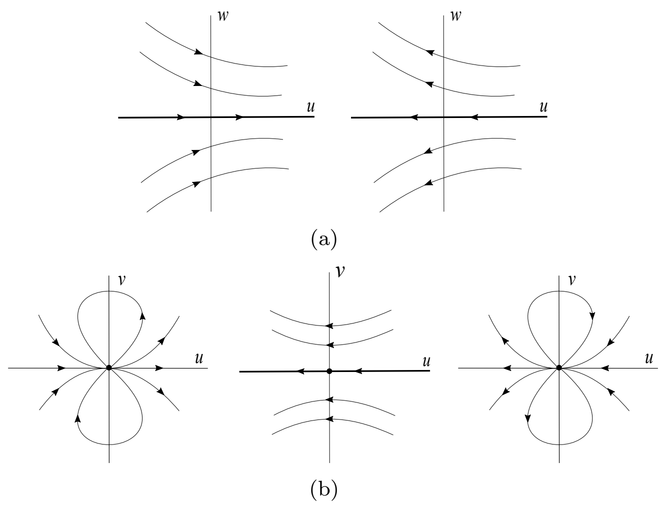


FIGURE 5. Blow-up at the origin of (28) with $a_{21} \neq 0$. (a) Systems (29) with $a_{21} > 0$ and $a_{21} < 0$ respectively; (b) Systems (28) with $a_{21} > 0$, $-1 < a_{21} < 0$ and $a_{21} < -1$ respectively.

(II.1) *If* $a_{21} > 0$ *or* $a_{21} < -1$ *the local phase portrait at the origin of systems* (28) *consists of two parabolic sectors and two elliptic sectors, see the first or the third picture in Figure 5(b), respectively.*
(II.2) *If* $-1 < a_{21} \leq 0$ *the local phase portrait at the origin of systems* (28) *consists of two hyperbolic sectors, see the second picture in Figure 5(b).*

Proof. Obviously the origin of systems (28) is not an infinite equilibrium point when $a_{03} \neq 0$.

If $a_{03} = 0$ the origin of systems (28) is a degenerate equilibrium point. In order to study its local phase portrait we apply the directional blow-up $(u, v) \to (u, w)$ with $w = v/u$ and the rescaling of the time $\mathrm{d}t = \mathrm{d}\tau/u^2$, then systems (28) become

$$u' = 2a_{21} - u^2 - w^2 - a_{21}w^2 + u^2w^2, \qquad w' = w(-a_{21} + w^2 + a_{21}w^2). \tag{29}$$

When $a_{21} \neq 0$ systems (29) have no equilibrium points for $u = 0$, therefore we study the local phase portrait in a neighborhood of the origin. Since $u'|_{w=0} = 2a_{21} - u^2$ and $w'|_{u=0} = -a_{21}w + O(w^3)$, the local phase portrait at the origin of (29) is given by Figure 5(a) when $a_{21} > 0$ and $a_{21} < 0$. Going back through the change of variables until systems (28) and taking into account the flow of systems (28) on the axes we have $\dot{u}|_{u=0} = -(1 + a_{21})v^2$, $\dot{u}|_{v=0} = 2a_{21}u^2 + O(u^4)$ and $\dot{v}|_{u=0} = 0$. Then we obtain that the local phase portrait at the origin of the chart U_2 consists of two parabolic and two elliptic sectors when $a_{21} > 0$ or $a_{21} < -1$, and it consists of two hyperbolic sectors when $-1 < a_{21} < 0$, see Figures 5(b).

When $a_{21} = 0$ systems (29) have one equilibrium point $(0, 0)$ for $u = 0$, which is degenerate. Again we apply the blow-up $(u, w) \to (u, V)$ with $V = w/u$ and obtain

$$u' = -1 - V^2 + u^2V^2, \qquad V' = V(1 + V^2 + uV^2 - u^2V^2), \tag{30}$$

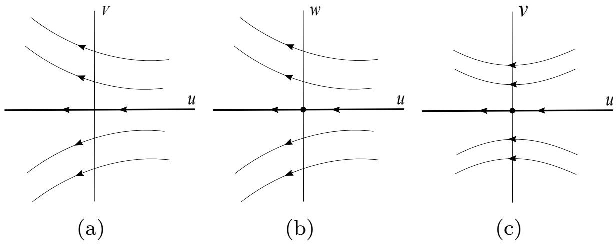


FIGURE 6. Blow-up at the origin of (28) with $a_{21} = 0$. (a) System (30); (b) Systems (29); (c) Systems (28).

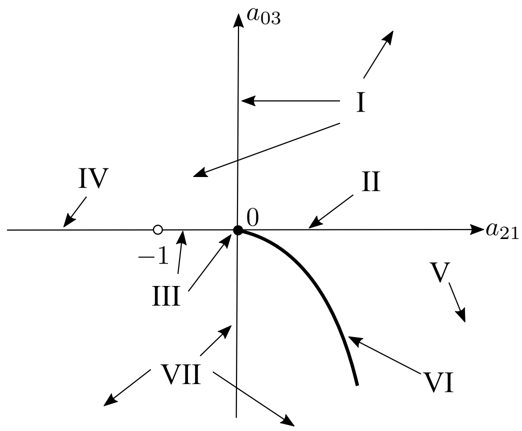


FIGURE 7. The bifurcation diagram of the infinite equilibrium points of systems (17) with $a_{03}^2 + (a_{21} + 1)^2 \neq 0$.

by a rescaling of the time $\mathrm{d}t = \mathrm{d}\tau/u^2$. Taking $u = 0$ system (30) has no equilibrium points. We reconstruct the flow through these changes of variables to obtain the local phase portrait of the origin of systems (28), and obtain that it consists of two hyperbolic sectors, see Figure 6. □

In summary the polynomial differential systems (17) have at most eight equilibrium points at infinity. And the bifurcation diagram of these infinite equilibria is shown in Figure 7, and the results for the infinite equilibria are listed in Table 1. Here we denote by C, S, N, H and E the center, the saddle, the node, the equilibrium point whose local phase portrait consists of two hyperbolic sectors, and the equilibrium point whose local phase portrait consists of two parabolic sectors and two elliptic sectors, respectively. Note that the degree of systems (17) is odd, the flow in the local chart V_i $(i = 1, 2)$ has the same sense as in the local chart U_i.

Now we shall study the finite equilibria of systems (17). These systems have the third finite equilibrium point $p_3 = (0, 0)$, which is a center or a saddle when $a_{21} < -1$ or $a_{21} > -1$, respectively. If $a_{21} = -1$ the origin p_3 is nilpotent. Doing

TABLE 1. The results for the infinite equilibrium points of systems (17) with $a_{03}^2 + (a_{21}+1)^2 \neq 0$.

Regions	Isolated infinite equilibria	Conditions
I	2 attracting N, 2 repelling N	$a_{03} > 0$
II	2 attracting N, 2 repelling N, 2 E	$a_{03} = 0$, $a_{21} > 0$
III	2 H	$a_{03} = 0$, $-1 < a_{21} \leq 0$
IV	2 E	$a_{03} = 0$, $a_{21} < -1$
V	4 attracting N, 4 repelling N	$a_{21} > 0$, $-a_{21}^2 < a_{03} < 0$
VI	4 H	$a_{21} > 0$, $a_{03} = -a_{21}^2$
VII	0	$a_{03} < 0$, $a_{21} < 0$ or $a_{21} > 0$, $a_{03} < -a_{21}^2$

the change of time $t \to -t$ systems (17) become

$$\begin{aligned} \dot{x} =& (x^2 - a_{03}y^2)y = H(x,y), \\ \dot{y} =& x(1 - x^2 - y^2) = x + G(x,y). \end{aligned} \tag{31}$$

Therefore $x = 0$ is one solution of $x + G(x,y) = 0$ in a neighborhood of the origin of systems (31). Then we have

$$H(0,y) = -a_{03}y^3$$

and

$$\left(\frac{\partial H}{\partial x} + \frac{\partial G}{\partial y}\right)\bigg|_{(0,y)} = 0.$$

From Theorem 3.5 of [15] we have that p_3 is a center or a saddle when $a_{03} > 0$ or $a_{03} < 0$, respectively.

In addition if $(1 + a_{21})a_{03} > 0$ systems (17) have the fourth and the fifth finite equilibrium points $p_{4,5} = (0, \pm\sqrt{(1 + a_{21})a_{03}}/a_{03})$, which are two centers or two saddles when $a_{03} + a_{21} + a_{21}^2 > 0$ or $a_{03} + a_{21} + a_{21}^2 < 0$, respectively. And they are nilpotent when $a_{03} + a_{21} + a_{21}^2 = 0$. We again apply Theorem 3.5 of [15] for these nilpotent equilibria and obtain that they are two saddles.

Now we consider the equilibrium points which are not on the x-axis or on the y-axis. Solving $-1 - a_{21} + a_{21}x^2 + a_{03}y^2 = 0$ and $-1 + x^2 - a_{21}y^2 = 0$, we obtain four possible equilibria

$$(x_i, y_i) = \left(\pm\frac{\sqrt{(a_{03} + a_{21}^2)(a_{03} + a_{21} + a_{21}^2)}}{a_{03} + a_{21}^2}, \pm\frac{\sqrt{a_{03} + a_{21}^2}}{a_{03} + a_{21}^2}\right),$$

for $i = 6,7,8,9$. Therefore systems (17) have four equilibrium points $p_i = (x_i, y_i)$ $(i = 6,7,8,9)$ different from the above five ones when $a_{03} + a_{21}^2 > 0$ and $a_{03} + a_{21} + a_{21}^2 > 0$, otherwise they have no these equilibrium points. And they are saddles because the traces and the determinants of the Jacobian matrix of (17) at $p_i = (x_i, y_i)$ $(i = 6,7,8,9)$ are zero and $-4(a_{03} + a_{21} + a_{21}^2)/(a_{03} + a_{21}^2) < 0$, respectively.

Proposition 4.3. *The phase portraits of systems* (17) *are topologically equivalent to*

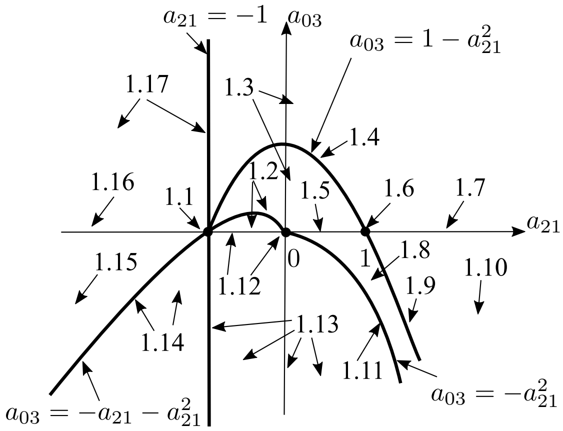


FIGURE 8. The bifurcation diagram of the phase portraits of systems (17).

1.1 *if* $a_{03} = 0$ *and* $a_{21} = -1$*;*
1.2 *if* $a_{03} > 0$*,* $a_{21} > -1$ *and* $a_{03} + a_{21} + a_{21}^2 \leq 0$*;*
1.3 *if* $a_{03} > 0$*,* $a_{21} > -1$*,* $a_{03} + a_{21} + a_{21}^2 > 0$ *and* $a_{03} \neq 1 - a_{21}^2$*;*
1.4 *if* $a_{03} > 0$*,* $a_{21} > -1$*,* $a_{03} + a_{21} + a_{21}^2 > 0$ *and* $a_{03} = 1 - a_{21}^2$*;*
1.5 *if* $a_{03} = 0$ *and* $0 < a_{21} < 1$*;*
1.6 *if* $a_{03} = 0$ *and* $a_{21} = 1$*;*
1.7 *if* $a_{03} = 0$ *and* $a_{21} > 1$*;*
1.8 *if* $0 > a_{03} > -a_{21}^2$*,* $a_{21} > 0$ *and* $a_{03} < 1 - a_{21}^2$*;*
1.9 *if* $0 > a_{03} > -a_{21}^2$*,* $a_{21} > 0$ *and* $a_{03} = 1 - a_{21}^2$*;*
1.10 *if* $0 > a_{03} > 1 - a_{21}^2$ *and* $a_{21} > 0$*;*
1.11 *if* $a_{03} = -a_{21}^2$ *and* $a_{21} > 0$*;*
1.12 *if* $a_{03} = 0$ *and* $0 \geq a_{21} > -1$*;*
1.13 *if* $a_{03} < -a_{21}^2$ *and* $a_{21} > 0$*, or* $a_{03} < 0$ *and* $-1 \leq a_{21} \leq 0$*;*
1.14 *if* $a_{03} \leq -a_{21} - a_{21}^2$ *and* $a_{21} < -1$*;*
1.15 *if* $0 > a_{03} > -a_{21} - a_{21}^2$ *and* $a_{21} < -1$*;*
1.16 *if* $a_{03} = 0$ *and* $a_{21} < -1$*;*
1.17 *if* $a_{03} > 0$ *and* $a_{21} \leq -1$*.*

Moreover the corresponding bifurcation diagram is shown in Figure 8.

Proof. We just prove the phase portraits from 1.3 to 1.10 because the proof of the others follows from the previous analysis and Propositions 4.1 and 4.2 directly.

1) If $a_{21} > -1$ and $a_{03} + a_{21} + a_{21}^2 > 0$ we obtain that systems (17) have four infinite equilibrium points on the Poincaré sphere by Propositions 4.1 and 4.2. The equilibrium points $A_{1,2}$ are in U_1, and the corresponding points in V_1, which are all nodes. And there are nine finite equilibrium points, where $p_{1,2,4,5}$ are four centers and $p_{3,6,7,8,9}$ are five saddles. Hence systems (17) have two bi-centers, one at $p_{1,2}$ and one at $p_{4,5}$. We have that the local phase portraits at these equilibria in the Poincaré disc are shown in Figure 9(a). And the Hamiltonian $H(x, y)$ has the same value at saddles $p_{6,7,8,9}$ because it is an even function.

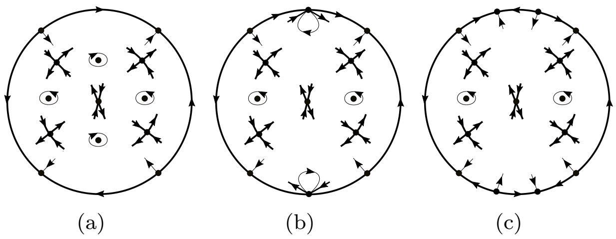


FIGURE 9. The local phase portraits at all equilibria of systems (17). (a) If $a_{21} > -1$ and $a_{03} + a_{21} + a_{21}^2 > 0$; (b) If $a_{03} = 0$ and $a_{21} > 0$; (c) If $0 > a_{03} > -a_{21}^2$ and $a_{21} > 0$.

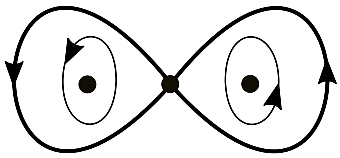


FIGURE 10. A eight–figure loop.

Since the finite equilibrium points are either centers or saddles, there must be at least one saddle on the boundary of the period annulus of each center. If only saddle p_3 is on the boundary of period annulus of p_1, taking into account the symmetry, p_3 must be also on the boundary of period annulus of p_2. Thus it creates one eight–figure loop, see Figure 10. Then there are two saddles on the boundary of period annulus of p_4. We suppose that these two saddles are $p_{6,8}$. So the other two saddles $p_{7,9}$ are on the boundary of period annulus of p_5. Furthermore the phase portraits of systems (17) in this case are topologically equivalent to the 1.3 of Figure 1.

If two saddles p_6 and p_7 are on the boundary of period annulus of p_1. Taking into account the symmetry the saddles p_8 and p_9 must be on the boundary of period annulus of p_2. Then the saddle p_3 must be on the boundary of the period annulus of the period annulus of p_4, by the symmetry, p_3 is also on the boundary of period annulus of p_5. And it creates one eight–figure loop. Hence in this case the global phase portrait in the Poincaré disc is shown in Figure 11. Since the separatrix skeletons of the phase portrait in Figure 11 and of the phase portrait 1.3 are equivalent, these two phase portraits are topologically equivalent.

From the phase portrait 1.3 to the phase portrait of Figure 11, and the continuity of the phase portraits with respect to the parameters there is a bifurcation curve $a_{03} = f(a_{21})$ which corresponds to the phase portrait 1.4 of Figure 1. In order to obtain this phase portrait it requires to find a set of explicit values a_{03} and a_{21}, which is not an easy task. But in this case we can find this bifurcation curve by computing the invariant straight lines of systems (17). In fact, assuming that $x = ky$ is an invariant line of systems (17), we have

$$\begin{aligned}\dot{x} - k\dot{y} &= (-1 - a_{21} + a_{21}k^2y^2 + a_{03}y^2)y - k^2y(-1 + k^2y^2 - a_{21}y^2) \\ &= (-1 - a_{21} + k^2)y + (a_{03} + 2a_{21}k^2 - k^4)y^3.\end{aligned} \tag{32}$$

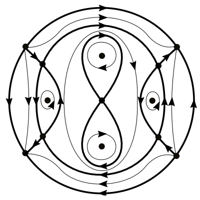


FIGURE 11. The phase portrait of systems (17) with $a_{21} > -1$, $a_{03} + a_{21} + a_{21}^2 > 0$ and $a_{03} > 1 - a_{21}^2$.

Hence when (32) is identically zero we have $k = \pm\sqrt{1 + a_{21}}$ and $a_{03} = 1 - a_{21}^2$. Furthermore the saddles $p_{6,9}$ are on the invariant straight line $x = \sqrt{1 + a_{21}}y$ and the saddles $p_{7,8}$ are on the invariant straight line $x = -\sqrt{1 + a_{21}}y$. And we have that the infinite equilibrium points $A_{1,2} = (y/x, 0) = (\pm\sqrt{1 + a_{21}}/(1 + a_{21}), 0)$ are on the lines $x = \sqrt{1 + a_{21}}y$ and $x = -\sqrt{1 + a_{21}}y$, respectively. Then there are three saddles p_3, p_6 and p_7 on the boundary of period annulus of p_1. In fact we obtain that $H(x, y)$ is zero at these three saddles. By the symmetry we obtain the phase portrait 1.4 of Figure 1.

2) Assume that $a_{03} = 0$ and $a_{21} > 0$, by Propositions 4.1 and 4.2, the infinite equilibrium points are $(\pm\sqrt{2a_{21}}/2a_{21}, 0)$ in U_1 and the origin in U_2, and also the corresponding points in $V_{1,2}$. The four infinite equilibrium points in U_1 and V_1 are nodes. The phase portraits at the origins of U_2 and V_2 both consist of two parabolic and two elliptic sectors. And systems (17) have seven finite equilibrium points, where $p_{1,2}$ are two centers and $p_{3,6,7,8,9}$ are five saddles. We have that the local phase portraits at these equilibria in Poincaré disc are shown in Figure 9(b).

If only the saddle p_3 is on the boundary of period annulus of p_1, taking into account the symmetry, this saddle is also on the boundary of period annulus of p_2. It creates one eight–figure loop. Then one attracting separatrix and one repelling separatrix of saddle p_6 connect with one repelling separatrix of the saddle p_7 and one attracting separatrix of the saddle p_8, respectively. By the symmetry we have that the phase portrait in the Poincaré disc is topologically equivalent to 1.5 of Figure 1, which is realized when $a_{21} = 0.5$.

In a similar way to the analysis of the phase portrait 1.4, we have that if $a_{03} = 1 - a_{21}^2$, i.e. $a_{21} = 1$ the straight lines $x = \pm\sqrt{2}y$ are two invariant lines of systems (17). Then the saddles $p_{6,9}$ and $p_{7,8}$ are on the invariant lines $x = \sqrt{2}y$ and $x = -\sqrt{2}y$, respectively. And the infinite equilibrium points $(\pm\sqrt{2}/2, 0)$ are also on these two invariant lines, respectively. Hence the saddles p_3, p_6 and p_7 are on the boundary of the period annulus of p_1. Taking into account the symmetry the phase portraits of systems (17) are topologically equivalent to 1.6 of Figure 1.

If the two saddles p_6 and p_7 are on the boundary of the period annulus of p_1. The separatrices of the saddle p_3 must go to the infinity. Then this phase portrait is topologically equivalent to 1.7 of Figure 1. When $a_{21} = 2$ this phase portrait is realized.

Moreover we have that the phase portraits of systems (17) in the Poincaré disc are topologically equivalent to 1.5, 1.6 and 1.7 of Figure 1 when $0 < a_{21} < 1$, $a_{21} = 1$ and $a_{21} > 1$, respectively.

3) Assume that $0 > a_{03} > -a_{21}^2$ and $a_{21} > 0$, the infinite equilibrium points are $A_{1,2,3,4}$ in U_1, and also the corresponding points in V_1. And all of them are nodes. In addition there are seven finite equilibrium points for systems (17), where $p_{1,2}$ are two centers and $p_{3,6,7,8,9}$ are five saddles. We obtain that the local phase portraits at these equilibria in the Poincaré disc are shown in Figure 9(c). Similarly to the above cases now the phase portraits of systems (17) are topologically equivalent to 1.8, 1.9 and 1.10 of Figure 1 when $a_{03} < 1 - a_{21}^2$, $a_{03} = 1 - a_{21}^2$ and $a_{03} > 1 - a_{21}^2$, respectively.

Further, all results for the isolated equilibrium points in the above cases are listed in Table 2. □

TABLE 2. The isolated equilibrium points corresponding to the topological phase portraits 1.1-1.17.

Phase portraits	Isolated finite equilibria	Isolated infinite equilibria
1.1	2 C	0
1.2	2 C, 3 S	2 attracting N, 2 repelling N
1.3,1.4	4 C, 5 S	2 attracting N, 2 repelling N
1.5,1.6,1.7	2 C, 5 S	2 attracting N, 2 repelling N, 2 E
1.8,1.9,1.10	2 C, 5 S	4 attracting N, 4 repelling N
1.11	2 C, 1 S	4 H
1.12	2 C, 1 S	2 H
1.13	2 C, 1 S	0
1.14	3 C, 2 S	0
1.15	5 C, 4 S	0
1.16	3 C, 4 S	2 E
1.17	3 C, 4 S	2 attracting N, 2 repelling N

4.2. **Phase portraits when $b_{03} > 0$.** Without loss of generality we can assume that $b_{03} = 1$ and systems (3) become

$$\begin{aligned} \dot{x} =& -(1 + a_{21})y + a_{21}x^2y - 3xy^2 + a_{03}y^3, \\ \dot{y} =& -x + x^3 - a_{21}xy^2 + y^3. \end{aligned} \tag{33}$$

Then we obtain that systems (33) have finitely many equilibrium points. Indeed, by computing the resultant of $\dot{x}$ and $\dot{y}$ with respect to x we have the polynomial

$$(-1 - a_{21})y^3 + F_1y^5 + F_2y^7 + F_3y^9, \tag{34}$$

where

$$\begin{aligned} F_1 =& 9 + 3a_{03} + 9a_{21} + 2a_{03}a_{21} + 2a_{21}^2 + 2a_{21}^3, \\ F_2 =& -9a_{03} - 3a_{03}^2 + 18a_{21} - a_{03}^2a_{21} + 15a_{21}^2 - 4a_{03}a_{21}^2 \\ & - 2a_{03}a_{21}^3 - a_{21}^4 - a_{21}^5, \\ F_3 =& 27 + a_{03}^3 - 18a_{03}a_{21} + 2a_{03}^2a_{21}^2 - 2a_{21}^3 + a_{03}a_{21}^4. \end{aligned} \tag{35}$$

For polynomial (34) to be identically zero, it needs $a_{21} = -1$ so that the coefficient of y^3 is zero. Then (34) simplifies to

$$a_{03}y^5 + (-3 - 11a_{03} - 2a_{03}^2)y^7 + (29 + 19a_{03} + 2a_{03}^2 + a_{03}^3)y^9. \tag{36}$$

It is easy to see that the coefficients of y^5 and y^7 in polynomial (36) cannot be zero at the same time. Hence polynomial (34) is not identically zero.

On the other hand, the Jacobian matrix of systems (33) at one finite equilibrium point (x, y) is

$$\begin{pmatrix} (2a_{21}x - 3y)y & M_1 \\ M_2 & -(2a_{21}x - 3y)y \end{pmatrix}, \tag{37}$$

where

$$M_1 = -1 - a_{21} + a_{21}x^2 - 6xy + 3a_{03}y^2, \qquad M_2 = -1 + 3x^2 - a_{21}y^2.$$

We claim that there is no degenerate finite equilibrium points for systems (33). Indeed, we obtain that $(2a_{21}x - 3y)y$ and M_2 have no common solutions, because the Gröbner basis for the polynomials $\dot{x}$, $\dot{y}$, $(2a_{21}x - 3y)y$ and M_2 is 1. We again calculate the Gröbner basis for $\dot{x}$, $\dot{y}$, $(2a_{21}x - 3y)y$ and M_1, then we obtain four polynomials $1 + a_{21}$, $(9 + 4a_{03})y^2$, $y^2(-12 + 31y^2)$ and $8x + 31y^3$. It means that if $a_{21} = -1$ the origin p_3 is nilpotent, and if $a_{21} = -1$ and $a_{03} = -9/4$ systems (33) have two nilpotent equilibrium points $(3\sqrt{3/31}, \pm 2\sqrt{3/31})$ additional to the origin. By Theorem 3.5 of [15] we have that the origin is a saddle and $(3\sqrt{3/31}, \pm 2\sqrt{3/31})$ are two cusps. Moreover the other finite equilibrium points must be saddles or centers.

Now we study the infinite equilibrium points of systems (33). In U_1 systems (33) become

$$\begin{aligned} \dot{u} &= 1 - 2a_{21}u^2 + 4u^3 - a_{03}u^4 - v^2 + u^2v^2 + a_{21}u^2v^2, \\ \dot{y} &= uv(-a_{21} + 3u - a_{03}u^2 + v^2 + a_{21}v^2). \end{aligned} \tag{38}$$

And the Jacobian matrix of systems (38) at $(u, 0)$ is

$$\begin{pmatrix} -4u(a_{21} - 3u + a_{03}u^2) & 0 \\ 0 & -u(a_{21} - 3u + a_{03}u^2) \end{pmatrix}. \tag{39}$$

Obviously the origin is not an equilibrium point of systems (38). And if $a_{03} = 0$ systems (38) have at most three equilibrium points on $v = 0$, and if $a_{03} \neq 0$ they have at most four equilibrium points on $v = 0$. From (39) it is easy to compute that these infinite equilibrium points in U_1 must be nodes or degenerate equilibria.

If

$$f_1(u) = 1 - 2a_{21}u^2 + 4u^3 - a_{03}u^4, \qquad f_2(u) = a_{21} - 3u + a_{03}u^2. \tag{40}$$

we compute the Gröbner basis for $f_1(u)$ and $f_2(u)$, then we obtain one polynomial

$$12a_{03} - a_{03}^2a_{21} - a_{21}^2 - a_{03}a_{21}^3 + (3a_{21} - 4a_{03}^2)u. \tag{41}$$

Hence systems (38) have at most one degenerate infinite equilibrium point for the value of u which vanishes the previous polynomial.

In order to provide a complete analysis of the global phase portraits of (33), it is necessary to study separately in $a_{03} = 0$ and $a_{03} \neq 0$.

4.2.1. *Subcase* $a_{03}=0$. By the above analysis we obtain that there are one and three infinite equilibrium points in U_1 when $a_{21}<3/\sqrt[3]{2}$ and $a_{21}>3/\sqrt[3]{2}$, respectively. And they are all nodes. On the other hand, if $a_{21}=3/\sqrt[3]{2}$ systems (36) have two infinite equilibrium points $(-1/(2\sqrt[3]{2}),0)$ and $(1/\sqrt[3]{2},0)$, which are a repelling node and a degenerate equilibrium point. We apply the blow-up technique and obtain that the local phase portrait of $(1/\sqrt[3]{2},0)$ consists of two hyperbolic sectors, as it is shown in Figure 6(c).

We proceed to analyze the origin of the local chart U_2. Systems (33) in this chart acquire the form

$$\begin{aligned}\dot u =& -4u+2a_{21}u^2-u^4-v^2-a_{21}v^2+u^2v^2,\\ \dot v =& v(-1+a_{21}u-u^3+uv^2).\end{aligned} \tag{42}$$

Obviously the origin of systems (42) is an equilibrium point, which is an attracting node.

We will now study the finite equilibria. We get that the third equilibrium point of systems (33) is $p_3=(0,0)$, which is a center or a saddle when $a_{21}<-1$ or $a_{21}>-1$, respectively. If $a_{21}=-1$, p_3 is a nilpotent saddle.

Next we consider the finite equilibria different from $p_{1,2,3}$. We compute the Gröbner basis for $-1-a_{21}+a_{21}x^2-3xy$ and $-x+x^3-a_{21}xy^2+y^3$, and obtain four polynomials, the following two

$$\begin{aligned}&(4a_{21}+3a_{21}^2)x+(3+12a_{21}+9a_{21}^2+a_{21}^3+a_{21}^4)y+9xy^2\\ &+(16a_{21}^2+15a_{21}^3-a_{21}^5-a_{21}^6)y^3+(27a_{21}-2a_{21}^4)y^5=0\end{aligned} \tag{43}$$

and

$$\begin{aligned}f(y)=&1+a_{21}+(-9-9a_{21}-2a_{21}^2-2a_{21}^3)y^2+(-18a_{21}-15a_{21}^2\\ &+a_{21}^4+a_{21}^5)y^4+(-27+2a_{21}^3)y^6=0,\end{aligned} \tag{44}$$

will be enough for our analysis. Then we obtain that there are at most six finite equilibrium points additional to $p_{1,2,3}$. But the explicit expressions of these finite equilibria and their eigenvalues in terms of parameter a_{21} are complicated, it is hard to study their existence and their types. Hence we need to present more algebraic tools for solving this problem.

Now we provide an useful way to obtain the information about the number of real roots of an arbitrary polynomial $f(x)$. We need to calculate the *discriminant sequence* $\{S_1,S_2,\cdots,S_n\}$ of $f(x)$ (for more details see [10, 35]) and determine its *sign list*

$$[\mathrm{sign}(S_1),\mathrm{sign}(S_2),\cdots,\mathrm{sign}(S_n)],$$

where the sign function is

$$\mathrm{sign}(x) = \begin{cases}-1, & \text{if } x<0,\\ 0, & \text{if } x=0,\\ 1, & \text{if } x>0.\end{cases} \tag{45}$$

For any sign list (SL) $[l_1,l_2,\cdots,l_n]$, we obtain the *revised sign list* (RSL) $[r_1,r_2,\cdots,r_n]$ as follows:

1. If $l_k\neq 0$ we write $r_k=l_k$.

2. If a section $[l_i, l_{i+1}, \cdots, l_{i+j}]$ of the given sign list satisfies with $l_{i+1} = \cdots = l_{i+j-1} = 0$ and $l_i l_{i+j} \neq 0$, we replace the subsection $[l_{i+1}, l_{i+2}, \cdots, l_{i+j-1}]$ with $[-l_i, -l_i, l_i, l_i, -l_i, -l_i, l_i, l_i, -l_i, \cdots]$ keeping the number of terms.

Then we obtain that the RSL $[r_1, r_2, \cdots, r_n]$ has no zeros between two nonzero members. From [35], we have the following theorem for computing the number of real roots of $f(x)$.

Theorem 4.4. *For any polynomial*

$$f(x) = a_0 + a_1 x + \cdots + a_k x^k \tag{46}$$

with real coefficients, if the number of the sign changes of the RSL $[r_1, r_2, \cdots, r_n]$ *of* (46) *equals* m*, and the number of nonzero elements of this RSL equals* n*, the number of the distinct real roots of* (46) *is* $n - 2m$.

Thus we can determine the number and the type of the remaining finite equilibrium points by Theorems 4.4, 2.1 and 2.3. And we do not need to calculate the coordinates of these equilibrium points.

Assume that $-27 + 2a_{21}^3 \neq 0$ and $1 + a_{21} \neq 0$, we compute the discriminant sequence $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ of $f(y)$ from (44), and obtain

$$\begin{array}{ll} S_1 = 6M_1^2, & S_2 = -12a_{21}M_1^3M_2, \\ S_3 = -16a_{21}M_1^3M_2M_3, & S_4 = -32(1+a_{21})M_1^3M_3M_4, \\ S_5 = -32(1+a_{21})^2M_1^3M_4M_5^2M_6, & S_6 = -64(1+a_{21})^3M_1^3M_5^4M_6^2, \end{array} \tag{47}$$

where

$$\begin{aligned} M_1 &= -27 + 2a_{21}^3, \qquad M_2 = -18 - 15a_{21} + a_{21}^3 + a_{21}^4, \\ M_3 &= -729 - 729a_{21} + 162a_{21}^2 + 432a_{21}^3 + 279a_{21}^4 - 24a_{21}^5 - 54a_{21}^6 \\ &\quad - 30a_{21}^7 + a_{21}^8 + 2a_{21}^9 + a_{21}^{10}, \\ M_4 &= -8748 - 10206a_{21} - 2187a_{21}^2 + 1620a_{21}^3 + 3078a_{21}^4 + 783a_{21}^5 \\ &\quad + 144a_{21}^6 - 316a_{21}^7 - 97a_{21}^8 - 42a_{21}^9 + 11a_{21}^{10} + 4a_{21}^{11} + 2a_{21}^{12}, \\ M_5 &= -27 - 27a_{21} + 4a_{21}^3 + 3a_{21}^4, \\ M_6 &= -135 - 81a_{21} - 45a_{21}^2 + 13a_{21}^3 + 8a_{21}^4 + 4a_{21}^5. \end{aligned} \tag{48}$$

From the above analysis of the infinite equilibrium points of systems (33), we separate the study in the following three cases.

1) If $a_{21} > 3/\sqrt[3]{2}$ we have $S_1 > 0$, $S_4 < 0$, $S_5 < 0$, $S_6 < 0$ and $S_2S_3 \geq 0$. We denote by $\mathrm{Root}[f(\alpha), i]$ the i-th real root of the polynomial $f(\alpha)$ with respect to α, and it has $\mathrm{Root}[f(\alpha), i] < \mathrm{Root}[f(\alpha), j]$ when $i < j$. We have $S_{2,3} > 0$ when $3/\sqrt[3]{2} < a_{21} < \mathrm{Root}[M_2, 2]$. And we have $S_{2,3} = 0$ when $a_{21} = \mathrm{Root}[M_2, 2]$, and $S_{2,3} < 0$ when $a_{21} > \mathrm{Root}[M_2, 2]$. Therefore the corresponding RSL of the first case is $[1, 1, 1, -1, -1, -1]$, and the RSL of the last two cases is $[1, -1, -1, -1, -1, -1]$. Thus, by Theorem 4.4, systems (33) have four finite equilibrium points $p_{6,7,8,9}$ other than $p_{1,2,3}$, where $p_{1,2}$ are two centers and p_3 is a saddle.

On the other hand, systems (33) have three infinite equilibrium points $A_i = (u_i, 0)$ $(i = 1, 2, 3)$ in U_1, one in the U_2, and the corresponding points in $V_{1,2}$. Denote that $u_1 > u_2 > u_3$. The eight infinite equilibria are nodes. Hence, on the Poincaré sphere the sum of the indices of the known equilibria is 10. By Theorem 2.3, the

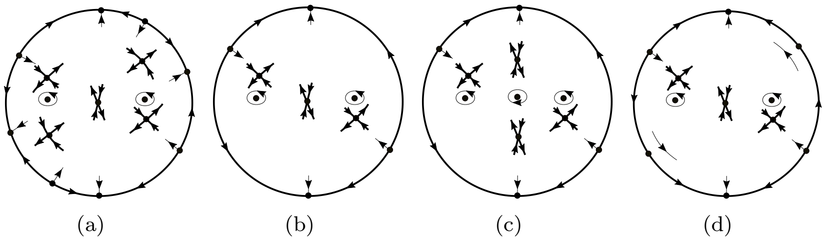


FIGURE 12. The local phase portraits at all equilibria of systems (33) with $a_{03} = 0$. (a) If $a_{21} > 3/\sqrt[3]{2}$; (b) If $-1 \le a_{21} < 3/\sqrt[3]{2}$; (c) If $a_{21} < -1$; (d) If $a_{21} = 3/\sqrt[3]{2}$.

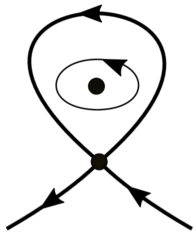


FIGURE 13. A center-loop.

sum of the indices of the remaining finite equilibria must be -8. Furthermore the remaining equilibrium points in the interior of the Poincaré disc are four saddles, where two must be symmetry to the other ones with respect to the origin. Then we obtain that the local phase portraits at these equilibria in the Poincaré disc are shown in Figure 12(a).

We claim that in this case systems (33) have at most two saddles on the boundary of period annulus of the bi-center at $p_{1,2}$. By computing the Gröbner basis for the three polynomials $\dot{x}/y$, $\dot{y}$ and $H(x, y) - h$ with some $h \in \mathbb{R}/\{0\}$, we get that $H(x, y)$ has the same non-zero values at no more than two equilibrium points. And we obtain four even polynomials $F_i(a_{21}, y, h)$ $(i = 1, 2, 3, 4)$ in y from this Gröbner basis, where $F_{2,3}(a_{21}, y, h)$ are quadratic and $F_4(a_{21}, y, h)$ is quartic. There is a polynomial $F_5(a_{21}, x, y, h)$ which is linear in x with nonzero coefficients. We calculate the resultant of the coefficients of y^2 in $F_2(a_{21}, y, h)$ and $F_3(a_{21}, y, h)$ with respect to h, and obtain $A(1+a_{21})M_6 > 0$, where A is a constant. Thus at least one coefficient of y^2 of the polynomials $F_{2,3}(a_{21}, y, h)$ is nonzero. It means that there are at most two distinct real roots for these three polynomials, then our claim is proved.

By the symmetry of systems (33) there must be only one saddle on the boundary of period annulus of each center, creating a center-loop, see Figure 13. Here we assume that these two saddles are $p_{7,8}$.

If one repelling and one attracting separatrix of the origin of systems (33) connect with the origin of U_2 and with the infinite equilibrium point A_1 of U_1, respectively, then we have that this phase portrait in the Poincaré disc is topologically equivalent

TABLE 3. The conditions of the revised sign lists of $f(y)$ when $-1 < a_{21} < 3/\sqrt[3]{2}$.

RSL	SL	Conditions
$[1,1,1,1,-1,1]$	$[1,1,1,1,-1,1]$	$-1 < a_{21} < \text{Root}[M_3,1]$;
$[1,1,-1,1,1,1]$	$[1,1,-1,1,1,1]$ $[1,1,-1,0,0,1]$	$\text{Root}[M_4,1] < a_{21} < 0$; $a_{21} = \text{Root}[M_4,1]$;
$[1,1,-1,-1,-1,1]$	$[1,1,-1,-1,-1,1]$ $[1,1,0,0,-1,1]$	$\text{Root}[M_3,1] < a_{21} < \text{Root}[M_4,1]$; $a_{21} = \text{Root}[M_3,1]$;
$[1,-1,-1,1,1,1]$	$[1,-1,-1,1,1,1]$ $[1,-1,-1,0,0,1]$	$\text{Root}[M_4,2] < a_{21} < 3/\sqrt[3]{2}$; $a_{21} = \text{Root}[M_4,2]$;
$[1,-1,-1,-1,1,1]$	$[1,-1,-1,-1,1,1]$	$\text{Root}[M_3,2] < a_{21} < \text{Root}[M_5,2]$ or $\text{Root}[M_5,2] < a_{21} < \text{Root}[M_6,1]$;
$[1,-1,-1,-1,-1,1]$	$[1,-1,-1,-1,-1,1]$	$\text{Root}[M_6,1] < a_{21} < \text{Root}[M_4,2]$;
$[1,-1,1,1,1,1]$	$[1,-1,1,1,1,1]$ $[1,-1,0,0,1,1]$	$0 < a_{21} < \text{Root}[M_3,2]$; $a_{21} = \text{Root}[M_3,2]$;
$[1,-1,-1,-1,0,0]$	$[1,-1,-1,-1,0,0]$	$a_{21} = \text{Root}[M_5,2]$ or $a_{21} = \text{Root}[M_6,1]$.

to 1.18 of Figure 1, which can be realized when $a_{21} = 4$. If one repelling and one attracting separatrix of the origin of systems (33) connect with the origin U_2 and with the infinite equilibrium point A_3 of U_1, respectively, then the phase portraits of systems (33) are topologically equivalent to 1.20 of Figure 1, which can be realized when $a_{21} = 2.5$.

From the phase portraits 1.18 to 1.20 it follows by the continuity of the phase portraits with respect to the parameters that there must exist one phase portrait that one attracting separatix of the origin of systems (33) connects with the finite equilibrium point p_6, which corresponds to the phase portrait 1.19 of Figure 1. In fact we can find this bifurcation point a_{21} by computing the invariant straight lines $x = ky$ of systems (33). Similarly to (32) we obtain $a_{21} = 3$ and $k = 2$. Then we have $p_{6,9} = (\pm 2\sqrt{6}/3, \pm\sqrt{6}/3)$ and the infinite equilibrium point $A_2 = (1/2, 0)$ are on the invariant line $x = 2y$. Moreover we obtain that the phase portrait in the Poincaré disc is topologically equivalent to 1.18 or 1.20 of Figure 1 when $a_{21} > 3$ or $3/\sqrt[3]{2} < a_{21} < 3$, respectively.

2) Assume that $-1 \le a_{21} < 3/\sqrt[3]{2}$. If $-1 < a_{21} < 3/\sqrt[3]{2}$ we obtain the possible RSL of $f(y)$ and their associated SL as it is shown in Table 3. By Theorem 4.4 systems (33) have two finite equilibrium points $p_{7,8}$ additional to $p_{1,2,3}$, where $p_{1,2}$ are two centers and p_3 is a saddle. Then on the Poincaré sphere the sum of the indices of the known equilibria is 6. From Theorem 2.3 the sum of the indices of the remaining equilibria must be -4. Thus, the two remaining finite equilibrium points $p_{7,8}$ are two saddles. The local phase portraits of systems (33) at these equilibria in the Poincaré disc are shown in Figure 12(b). On the other hand, if $a_{21} = -1$ we have $p_{7,8} = (\pm 3\sqrt{3/29}, \mp\sqrt{3/29})$, which also are two saddles. And the y-axis is invariant by the flow of systems (33). Similarly to the above case the saddles $p_{7,8}$

TABLE 4. The conditions of the revised sign lists of $f(y)$ when $a_{21} < -1$.

RSL	SL	Conditions
$[1, 1, 1, -1, -1, -1]$	$[1, 1, 1, -1, -1, -1]$	$\text{Root}[M_2, 1] < a_{21} < \text{Root}[M_5, 1]$ or $\text{Root}[M_5, 1] < a_{21} < -1$;
$[1, -1, -1, -1, -1, -1]$	$[1, -1, -1, -1, -1, -1]$	$a_{21} < \text{Root}[M_2, 1]$;
	$[1, 0, 0, -1, -1, -1]$	$a_{21} = \text{Root}[M_2, 1]$;
$[1, 1, 1, -1, 0, 0]$	$[1, 1, 1, -1, 0, 0]$	$a_{21} = \text{Root}[M_5, 1]$.

must be on the boundary of period annulus of the bi-center at $p_{1,2}$, respectively. And they create two center-loops. Then we have that the phase portraits of systems (33) with $-1 \le a_{21} < 3/\sqrt[3]{2}$ are topologically equivalent to 1.21 of Figure 1.

3) If $a_{21} < -1$ then the RSL of $f(y)$ and their associated SL are given in Table 4.

From Theorem 4.4, if $a_{21} \neq \text{Root}[M_5, 1]$ the polynomial $f(y)$ has four distinct real roots. Thus systems (33) have four finite equilibrium points $p_{6,7,8,9}$ additional to $p_{1,2,3}$, where $p_{1,2,3}$ are three centers. And systems (33) have four infinite equilibria, which are nodes. Hence the sum of the indices of the known equilibria is 10 on the Poincaré sphere. By Theorem 2.3, the sum of the indices of the remaining equilibria must be -8. Thus, they are four saddles in Poincaré disc. If $a_{21} = \text{Root}[M_5, 1]$ there two distinct roots and two repeated real roots for $f(y)$. In fact systems (33) also have four finite equilibrium points $p_{6,7,8,9}$ different from $p_{1,2,3}$, and they are four saddles. Then we have that the local phase portraits at all equilibria in the Poincaré disc are shown in Figure 12(c).

From the previous proof we have that $H(x, y)$ has the same non-zero values at no more than two equilibrium points. Hence there must be one saddle on the boundary of the period annulus of each center, and every one of these saddles creates a center-loop. Here we assume that these two previous saddles are $p_{7,8}$. And the other two saddles $p_{6,9}$ must be on the boundary of the period annulus of the center at the origin. Then the phase portraits of systems (33) in the Poincaré disc are topologically equivalent to 1.22 of Figure 1.

Now we consider the case $a_{21} = 3/\sqrt[3]{2}$, then systems (33) have two finite equilibrium points $p_{7,8} \approx (\pm 1.06093, \mp 0.22024)$ additional to $p_{1,2,3}$, and they are two saddles. And systems (33) have six infinite equilibrium points, three in $U_{1,2}$ and three in $V_{1,2}$. We have that the local phase portraits of all equilibria in the Poincaré disc are shown in Figure 12(d). Similarly to the above case, there are no more than two equilibrium points in the same energy level. Hence the saddles $p_{7,8}$ must be on the boundary of the period annulus of the centers $p_{1,2}$, respectively. Then these two saddles create two center-loops. Therefore the phase portrait of systems (33) now is topologically equivalent to 1.23 of Figure 1.

In summary all results for the isolated equilibrium points of the phase portraits 1.18-1.23 are listed in Table 5. Further, we have the following result.

TABLE 5. The isolated equilibrium points corresponding to the topological phase portraits 1.18-1.23.

Phase portraits	Isolated finite equilibria	Isolated infinite equilibria
1.18,1.19,1.20	2 C, 5 S	4 attracting N, 4 repelling N
1.21	2 C, 3 S	2 attracting N, 2 repelling N
1.22	3 C, 4 S	2 attracting N, 2 repelling N
1.23	2 C, 3 S	2 attracting N, 2 repelling N, 2 H

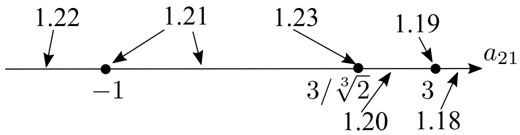


FIGURE 14. The bifurcation diagram of the phase portraits of systems (33) with $a_{03} = 0$.

Proposition 4.5. *The phase portraits of systems* (33) *in the Poincaré disc with* $a_{03} = 0$ *are topologically equivalent to*
1.18 *if* $a_{21} > 3$*;*
1.19 *if* $a_{21} = 3$*;*
1.20 *if* $3/\sqrt[3]{2} < a_{21} < 3$*;*
1.21 *if* $-1 \le a_{21} < 3/\sqrt[3]{2}$*;*
1.22 *if* $a_{21} < -1$*;*
1.23 *if* $a_{21} = 3/\sqrt[3]{2}$.
Moreover the bifurcation diagram is shown in Figure 14.

4.2.2. *Subcase* $a_{03} \neq 0$. In this subcase the characterization of the phase portraits of systems (33) become extremely difficult because the explicit expressions of some infinite and finite equilibrium points are complicated. To overcome these difficulties we do the following analysis.

We compute the Gröbner basis for $-1 - a_{21} + a_{21}x^2 - 3xy + a_{03}y^2$ and $-x + x^3 - a_{21}xy^2 + y^3$ from (33), and obtain five polynomials. And we only need to consider two polynomials, one is linear in x and the other one is

$$-1 - a_{21} + F_1 y^2 + F_2 y^4 + F_3 y^6, \tag{49}$$

where $F_{1,2,3}$ are shown in (35). Hence there are at most six finite equilibrium points additional to $p_{1,2,3}$. Then we compute the discriminant sequence $\{S_1, S_2, S_3, S_4\}$ of polynomial $f_1(u)$ from (40) for determining the infinite equilibrium points. And we have

$$\begin{aligned} &S_1 = 4a_{03}^2, \qquad S_2 = -16a_{03}^2(-3 + a_{03}a_{21}), \qquad S_3 = -64a_{03}^2 F_4, \\ &S_4 = -256a_{03}^2 F_3, \end{aligned} \tag{50}$$

where F_3 is shown in (35) and

$$F_4 = -3a_{03} + a_{03}^2 a_{21} - 2a_{21}^2 + a_{03}a_{21}^3.$$

We separate the study in the two cases $a_{21} \geq -1$ or $a_{21} < -1$, then p_3 is a saddle or a center, respectively.

1) Assume that $a_{21} \geq -1$.

1.a) When the polynomial $f_1(u)$ has four distinct real roots, the RSL of its discriminant sequence must be $[1, 1, 1, 1]$, which needs either $a_{03} < 0$ and $a_{21} > \text{Root}[F_3, 2]$, or $0 < a_{03} < 3/\sqrt[3]{16}$ and $\text{Root}[F_3, 1] < a_{21} < \text{Root}[F_3, 2]$. Then systems (38) have four equilibrium points in U_1, which are all nodes, and the origin of U_2 is not an equilibrium point.

The sum of the indices of the known equilibria is 10 on the Poincaré sphere. By Theorem 2.3, the sum of the indices of the remaining equilibria must be -8. From the previous analysis of (37) we have that the remaining finite equilibrium points are elementary and a pair of points are symmetric with respect to the origin. Thus they must be four saddles. Similarly to the subcase $a_{03} = 0$, by computing the Gröbner basis for $\dot{x}$, $\dot{y}$ and $H(x, y) - h$ we obtain that $H(x, y)$ has the same $h \in \mathbb{R}/\{0\}$ at no more than two finite equilibrium points. Then we obtain that the global phase portraits of systems (33) in the Poincaré disc are topologically equivalent to 1.18 and 1.20 of Figure 1. In particular when $a_{03} = 1 - a_{21}^2 + 4\sqrt{1 + a_{21}}$ the phase portrait is topologically equivalent to 1.19 of Figure 1, which has an invariant straight line. Indeed, forcing that the straight line $x = ky$ be this invariant line of systems (33) we obtain

$$\dot{x} - k\dot{y} = (-1 - a_{21} + k^2)y + (a_{03} + 2a_{21}k^2 - k^4 - 4k)y^3. \tag{51}$$

Hence we need that $G_1 = -1 - a_{21} + k^2 = 0$ and $G_2 = a_{03} + 2a_{21}k^2 - k^4 - 4k = 0$ in order that (51) be identically zero. The resultant of G_1 and G_2 with respect to the variable k is

$$-15 - 2a_{03} + a_{03}^2 - 16a_{21} - 2a_{21}^2 + 2a_{03}a_{21}^2 + a_{21}^4.$$

This resultant is zero if $a_{03} = 1 - a_{21}^2 + 4\sqrt{1 + a_{21}}$ or $a_{03} = 1 - a_{21}^2 - 4\sqrt{1 + a_{21}}$, but the latter cannot occur in this case. Indeed, we just need to substitute it into (50) and check its RSL.

1.b) When the polynomial $f_1(u)$ has three distinct real roots, the RSL of its discriminant sequence must be $[1, 1, 1, 0]$, which needs either $a_{03} < 3/\sqrt[3]{16}$ and $a_{21} = \text{Root}[F_3, 2]$, or $0 < a_{03} < 3/\sqrt[3]{16}$ and $a_{21} = \text{Root}[F_3, 1]$. From these two cases we have $F_3 = 0$, thus systems (33) have three infinite equilibrium points, where one is degenerate and the other two are nodes. Let $A_1 = (u_1, 0)$ be the degenerate infinite equilibrium point, and $A_2 = (u_2, 0)$ and $A_3 = (u_3, 0)$ be two nodes, for $u_1 > u_2 > u_3$. In addition the polynomial (49) becomes a quartic polynomial, hence there are at most four finite equilibrium points additional to $p_{1,2,3}$.

When $0 < a_{03} < 3/\sqrt[3]{16}$ and $a_{21} = \text{Root}[F_3, 1]$ the local phase portrait of the degenerate point A_1 consists of two hyperbolic sectors, see the second picture of Figure 5(b). By Theorem 2.3 the sum of the indices of the known equilibria of systems (33) is 6 on the Poincaré sphere. Then the sum of the indices of the remaining finite equilibria must be -4. By the symmetry of the equilibria the only possibility is that systems (33) have two saddles different from $p_{1,2,3}$ in the Poincaré disc. Then we have that in this case the phase portrait in the Poincaré disc is topologically equivalent to 1.23 of Figure 1, which is realized when $a_{03} = 1$ and $a_{21} = 2$.

When $a_{03} < 3/\sqrt[3]{16}$ and $a_{21} = \text{Root}[F_3, 2]$ the local phase portrait of A_1 consists of two parabolic and two elliptic sectors, see the first picture of Figure 5(b). Moreover the remaining finite equilibria must have total index -8. By the symmetry the only possibility is that they are four saddles. We have that the local phase portraits at all equilibria in the Poincaré disc are topologically equivalent to Figure 9(b), which it is not symmetric with respect to the x-axis or the y-axis. On the other hand there must be one saddle on the boundary of the period annulus of each center, creating two center-loops. We assume that these two saddles are $p_{7,8}$. If one repelling and one attracting separatrices of the origin connect to the infinite equilibrium point A_1, we obtain the phase portrait 1.24 of Figure 1, and when $a_{03} = 0.7$ and $a_{21} = 2.9198$ this phase portrait is achieved. If one repelling and one attracting separatrices of the origin connect to the infinite equilibrium points A_1 and A_3 respectively, we have the phase portrait 1.26 of Figure 2, which is realized when $a_{03} = 1$ and $a_{21} = 2.13488$.

From the phase portraits 1.24 and 1.26 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.25 of Figure 2, which has one invariant straight line. Indeed, by calculating $F_3 = G_1 = G_2 = 0$ from (51) we obtain either $a_{03} = 0.71794..$ and $a_{21} = 2.85173..$, or $a_{03} = -3$ and $a_{21} = 0$. But the latter is not satisfied in this subcase.

1.c) When the polynomial $f_1(u)$ has two distinct real roots, the RSL of its discriminant sequence must be either $[1, -1, -1, -1]$, or $[1, 1, -1, -1]$, or $[1, 1, 1, -1]$, or $[1, 1, 0, 0]$, which just needs either $F_3 > 0$, or $a_{21} = 3/\sqrt[3]{4}$ and $a_{03} = 3/\sqrt[3]{16}$. Then there are four infinite equilibria for systems (33), and all of them are nodes. Furthermore we have that the sum of the indices of the remaining finite equilibria on the Poincaré sphere must be -4. By (37) we know that the remaining finite equilibria are elementary. Then additional to the equilibria $p_{1,2,3}$, there are either (i) two saddles $p_{7,8}$, or (ii) two centers $p_{4,5}$ and four saddles $p_{6,7,8,9}$.

In case (i) if the two saddles $p_{7,8}$ are on the boundary of the period annulus of the bi-center, creating two center-loops. In this case we deduce that the global phase portrait is topologically equivalent to 1.21 of Figure 1. If the saddle at the origin is located on the boundary of the period annulus of p_1, by the symmetry it is also in the boundary of the period annulus of p_2. When the saddles $p_{7,8}$ are outside the region enclosed by the separatrices of the origin the global phase portraits of systems (33) are topologically equivalent to 1.2 of Figure 1, which can be realized when $a_{21} = 2$ and $a_{03} = -10$.

By the continuity of the phase portraits with respect to parameters a_{03} and a_{21} from the phase portrait 1.2 to 1.21, we obtain the bifurcation curve $a_{03} = 1 - a_{21}^2 - 4\sqrt{1 + a_{21}}$, on this curve we have the phase portrait 1.27 of Figure 2, i.e. the saddles at origin and $p_{7,8}$ are on the invariant straight line $x = -\sqrt{1 + a_{21}}y$. This phase portrait can be realized when $a_{21} = 2$ and $a_{03} = -9.9282$. And the invariant straight line can be obtained from (51).

In case (ii) we have that the local phase portraits at all equilibrium points are shown in Figure 9(a). If only the two saddles $p_{7,8}$ and the two centers $p_{1,2}$ create

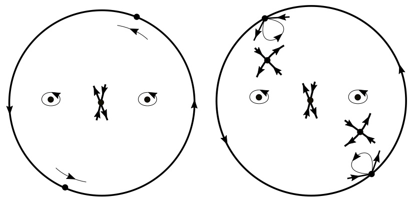


FIGURE 15. The local phase portraits at all equilibria of systems (33) with $a_{03} \neq 0$ and $a_{21} \geq -1$ in case 1.d). (a) Case (i); (b) Case (ii).

two center-loops, and if the origin p_3 is on the boundary of the period annulus of p_4, by the symmetry, the origin also will be on the boundary of the period annulus of p_5. Then it creates one eight–figure loop. Furthermore the origin must be inside the region enclosed by the separatrices of the saddles p_6 and p_9. Therefore the phase portrait is topologically equivalent to 1.28 of Figure 2, and this phase portrait can be obtained when $a_{03} = 2$ and $a_{21} = -0.9$. If there are four saddles $p_{6,7,8,9}$ located on the boundary of the period annulus of each center, creating four center-loops, then the phase portrait is topologically equivalent to 1.30 of Figure 2, which can be realized when $a_{03} = 2$ and $a_{21} = -0.8$.

From the phase portraits 1.28 and 1.30 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.29 of Figure 2, which can be realized when $a_{21} = 1$ and $a_{03} = 5.65685$. That is, the saddles $p_{3,6,9}$ are on one invariant straight line, and the saddles $p_{3,6}$ are on the boundary of the period annulus of p_4, by the symmetry the saddles $p_{3,9}$ are on the boundary of the period annulus of p_5. Similarly to subcase 1.a) this phase portrait can be obtained when $a_{03} = 1 - a_{21}^2 + 4\sqrt{1 + a_{21}}$ from (51).

1.d) When the polynomial $f_1(u)$ has one real root, the RSL of its discriminant sequence must be either $[1, 0, 0, 0]$, or $[1, 1, -1, 0]$, or $[1, -1, -1, 0]$, which needs $a_{03} \leq \text{Root}[29 + 19a_{03} + 2a_{03}^2 + a_{03}^3, 1] = -1.58141..$ and $a_{21} = \text{Root}[F_3, 1]$. From $F_3 = 0$ we obtain that systems (33) have one degenerate infinite equilibrium point in U_1. The phase portrait of this infinite equilibrium point consists of either (i) two hyperbolic sectors, or (ii) two parabolic sectors and two elliptic sectors because we have $\dot{u}|_{v=0} = 1 + O(u)$ and $\dot{v}|_{u=0} = 0$ from (38).

Assume (i). By Theorem 2.3 the sum of the indices of the known equilibria of (33) is 2 on the Poincaré sphere. Thus the finite equilibrium points of systems (33) different from $p_{1,2,3}$ can be either (i.a) no equilibrium points, or (i.b) two centers and two saddles, but the latter cannot occur. Indeed, when $F_3 = 0$ the polynomial (49) simplifies to

$$-1 - a_{21} + F_1 y^2 + F_2 y^4, \tag{52}$$

where $F_{1,2}$ are shown in (35). If we replace y^2 by z we get the quadratic polynomial

$$-1 - a_{21} + F_1 z + F_2 z^2. \tag{53}$$

Hence (52) has four real roots when (53) has two positive roots. Then we can use *Descartes' rule of signs* for studying the number of positive roots of (53) given in the following theorem.

Theorem 4.6. *The number of positive roots of a real polynomial* (46) *is either the number of sign differences between consecutive nonzero coefficients, or less than it by a multiple of* 2.

When (53) has two positive roots there must be two changes of sign between the coefficients of (53). Assume that $a_{21} > -1$ we have $-1 - a_{21} < 0$. Then the coefficients of z and z^2 must be positive and negative, respectively. Hence it must satisfy with $F_1 > 0$ and $F_2 < 0$, i.e. $0 < a_{03} < 3/\sqrt[3]{16}$ and $a_{21} = \text{Root}[F_3, 2]$, which is in contradiction with this case. Therefore systems (33) have at most two finite equilibria different from $p_{1,2,3}$.

Consider case (i.a). We obtain that the local phase portraits at all equilibria in the Poincaré disc are shown in Figure 15(a). If the origin is on the boundary of the period annulus of p_1, by the symmetry it is also on the boundary of the period annulus of p_2, creating one eight–figure loop. Then in this case the phase portrait is topologically equivalent to 1.12 of Figure 1, which can be realized when $a_{03} = -10$ and $a_{21} = 1.98273$. If the origin is not on the boundary of the period annulus of the bi-center at $p_{1,2}$, the separatrices of p_3 must connect to some infinite equilibrium points. And the bi-center at $p_{1,2}$ must be inside the region enclosed by the separatrices of the origin, then we have the phase portrait 1.31 of Figure 2. In fact we can obtain that p_3 and the infinite equilibria are on the invariant straight line $x = -y$, see (51). This phase portrait can be realized when $a_{03} = -3$ and $a_{21} = 0$.

Assume (ii). Then the sum of the indices of the remaining finite equilibria must be -4 on the Poincaré sphere. Hence systems (33) have two saddles $p_{7,8}$ different from $p_{1,2,3}$. The local phase portraits of systems (33) at these equilibria in Poincaré disc are shown in Figure 15(b). We claim that the saddles $p_{7,8}$ neither are connected with the origin, nor are outside the region enclosed by the boundary of the period annulus of p_3, as it is shown in the first three pictures of Figure 16. The first two pictures of Figure 16 cannot occur, otherwise there would have six intersection points with the separatrices of $p_{7,8}$ on a straight line l through p_3. That is, there would be six points on l with same energy level, but it is not possible for a quartic Hamiltonian $H(x, y)$. In the third picture of Figure 16 the flow at infinity is clockwise, in contradiction with the fact that $\dot{u}|_{u=0,v=0} = 1 > 0$ at the origin of the chart U_1. Then the phase portrait in Poincaé disc is topologically equivalent to 1.32 of Figure 2. This phase portrait can be realized by taking $a_{03} = -2$ and $a_{21} = -0.61752$.

1.e) The polynomial $f_1(u)$ has no real roots if and only if either $S_2 > 0$, $S_3 \leq 0$ and $S_4 > 0$, or $S_2 \leq 0$ and $S_4 > 0$, i.e. $a_{03} < \text{Root}[29 + 19a_{03} + 2a_{03}^2 + a_{03}^3, 1]$ and $-1 \leq a_{21} < \text{Root}[F_3, 1]$. By Theorem 2.3, on the Poincaré sphere the known equilibrium points of (33) have total index 2. Then the sum of indices of the remaining finite equilibria must be 0. Thus we have the following possibilities: (i) no equilibrium points, (ii) two cusps, (iii) two centers and two saddles.

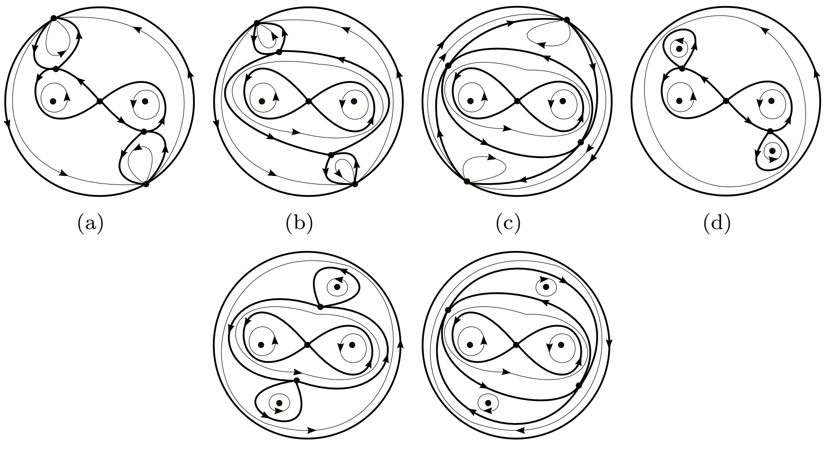


FIGURE 16. The possible phase portraits for the separatrices configurations of two saddles.

Case (i) can occur and we get that the phase portrait in the Poincaré disc is topologically equivalent to 1.13 of Figure 1, which can be realized when $a_{21} = -0.7$ and $a_{03} = -3$.

Consider case (iii), then the two additional saddles cannot be outside the region enclosed by the separatrices of the saddle p_3 and they cannot be connected with p_3. That is, the forth and fifth phase portraits of Figure 16 are not possible because a straight line through p_3 would intersect the separatrices of these two saddles six times. For the sixth picture of Figure 16 the flow at infinity is clockwise and this is not possible. Indeed, we have $\dot{u}|_{u=0,v=0} = 1 > 0$ at the origin of the chart U_1. Then we deduce that the phase portrait of systems (33) is topologically equivalent to 1.33 of Figure 2, which is achieved when $a_{21} = -0.7$ and $a_{03} = -2$. And it has two eight–figure loops due to the centers $p_{1,2}$ and the four remaining finite equilibrium points.

By the analysis of (37) when $a_{03} = -9/4$ and $a_{21} = -1$ systems (33) have two nilpotent equilibria $(3\sqrt{3/31}, \pm 2\sqrt{3/31})$, which are two cusps. Then we will obtain the phase portrait in case (ii). Since there is no more equilibrium points these cusps must be on the boundary of the period annulus of each center. Hence we have that the phase portrait in the Poincaé disc is topologically equivalent to 1.34 of Figure 2.

2) Assume that $a_{21} < -1$. The polynomial $f_1(u)$ has at most two distinct real roots because there are no parameters such that the RSL of its discriminant sequence is either $[1, 1, 1, 1]$, or $[1, 1, 1, 0]$. Since $a_{03} = -9/4$ and $a_{21} = -1$ are not in this case, the remaining finite equilibria are elementary, i.e. they are saddles or centers.

Similarly to the analysis of case 1.c) the polynomial $f_1(u)$ has two distinct real roots if and only if $S_4 < 0$, i.e. either $\mathrm{Root}[29 + 19a_{03} + 2a_{03}^2 + a_{03}^3, 1] < a_{03} < 0$ and $\mathrm{Root}[F_3, 1] < a_{21} < -1$, or $a_{03} > 0$ and $a_{02} < -1$. Then there are four infinite equilibria for systems (33), which are nodes. Hence on the Poincaré sphere the sum of the indices of the remaining finite equilibria must be -8. Thus there are four saddles additional to the equilibria $p_{1,2,3}$. Since the Hamiltonian $H(x, y)$ has the same non-zero value at no more than two equilibria, there are at most two saddles on the boundary of the period annulus of p_3. And by the symmetry the only way to have two centers and the two other saddles is to have two center-loops. Therefore the global phase portrait in the Poincaé disc is topologically equivalent to 1.22 of Figure 1.

Similarly to cases 1.d) and 1.e) we have the phase portrait 1.35 when $\text{Root}[29 + 19a_{03} + 2a_{03}^2 + a_{03}^3, 1] < a_{03} < 0$ and $a_{21} = \text{Root}[F_3, 1]$, and the phase portrait 1.14 or 1.36 when $a_{03} < 0$ and $a_{21} \leq \text{Root}[F_3, 1]$. On the other hand the phase portraits 1.14, or 1.35, or 1.36 can be realized when $a_{03} = -10$ and $a_{21} = -2$, or $a_{03} = -0.9491$ and $a_{21} = -2$, or $a_{03} = -2$ and $a_{21} = -2$, respectively.

Then, all results for the isolated equilibrium points of the phase portraits 1.24-1.36 are listed in Table 6.

Table 6. The isolated equilibrium points corresponding to the topological phase portraits 1.24-1.36.

Phase portraits	Isolated finite equilibria	Isolated infinite equilibria
1.24,1.25,1.26	2 C, 5 S	2 attracting N, 2 repelling N, 2 E
1.27	2 C, 3 S	2 attracting N, 2 repelling N
1.28,1.29,1.30	4 C, 5 S	2 attracting N, 2 repelling N
1.31	2 C, 1 S	2 H
1.32	2 C, 3 S	2 E
1.33	4 C, 3 S	0
1.34	2 C, 1 S, 2 cusp	0
1.35	3 C, 4 S	2 E
1.36	5 C, 4 S	0

Thus we have obtained all the phase portraits of the Z_2-equivariant cubic Hamiltonian systems (3) with a linear type bi-center, which are provided in Theorem 1.1.

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References

[1] W. Aziz, J. Llibre, C. Pantazi, *Centers of quasi-homogeneous polynomial differential equations of degree three*, Adv. Math. **254** (2014), 233–250.

[2] M.J. Álvarez, A. Ferragut, X. Jarque, *A survey on the blow up technique*, Int. J. Bifurcation Chaos **31** (2011), 3103–3118.

[3] J. Artés, J. Llibre, *Quadratic Hamiltonian vector fields*, J. Differential Equations **107** (1994), 80–95.

[4] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb. **30** (1952), 181–196; Amer. Math. Soc. Transl. **100** (1954), 1–19.

[5] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, *Local analytic integrability for nilpotent centers*, Ergodic Theory Dynam. Systems **23** (2003), 417–428.

[6] J. Chavarriga, J. Giné, *Integrability of a linear center perturbed by a fourth degree homogeneous polynomial*, Publ. Mat. **40** (1996,. 21–39.

[7] J. Chavarriga, J. Giné, *Integrability of a linear center perturbed by a fifth degree homogeneous polynomial*, Publ. Mat. **41** (1997), 335–356.

[8] T. Chen, L. Huang, P. Yu, W. Huang, *Bifurcation of limit cycles at infinity in piecewise polynomial systems*, Nonlinear Anal.: Real World Appl. **41** (2018), 82–106.

[9] T. Chen, W. Huang, D. Ren, *Weak center and local critical periods for a Z_2-equivariant cubic system*, Nonlinear Dyn. **78** (2014), 2319–2329.

[10] T. Chen, J. Llibre, *Bifurcation diagrams and global phase portraits for some Hamiltonian systems with rational potentials*, Int. J. Bifurcation Chaos **32** (2018), 1850168 (32 pp).

[11] A. Cima, J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, J. Math. Anal. Appl. **47** (1990), 420–448.

[12] I. Colak, J. Llibre, C. Valls, *Hamiltonian linear type centers of linear plus cubic homogeneous polynomial vector fields*, J. Differential Equations **257** (2014), 1623–1661.

[13] I. Colak, J. Llibre, C. Valls, *Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields*, Adv. Math. **259** (2014), 655–687.

[14] H. Dulac, *Détermination et integration d'une certaine classe d'équations différentielle ayant par point singulier un centre*, Bull. Sci. Math. **32** (1908), 230–252.

[15] F. Dumortier, J. Llibre, J. Artés, *Qualitative Theory of Planar Differential Systems*, Universitext, Springer-Verlag, New York, 2006.

[16] A. Gasull, A. Guillamon, V. Mañosa, *Phase portrait of Hamiltonian systems with homogeneous nonlinearities*, Nonlinear Anal. **42** (2000), 679–707.

[17] H. Giacomini, J. Giné, J. Llibre, *The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems*, J. Differential Equations **227** (2006), 406–426.

[18] W. Kapteyn, *On the midpoints of integral curves of differential equations of the first degree*, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. Konikl. Nederland **19** (1911), 1446–1457 (in Dutch).

[19] W. Kapteyn, *New investigations on the midpoints of integrals of differential equations of the first degree*, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **20** (1912), 1354–1365; Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **21** (1913), 27–33 (in Dutch).

[20] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Int. J. Bifurcation Chaos **3** (2013), 47–106.

[21] J. Li, Y. Liu, *New results on the study of Z_q-equivariant planar polynomial vector fields*, Qual. Theo. Dyna. Syst. **9** (2010), 167–219.

[22] J. Li, M. Zhang, *Bifurcations of limit cycles of Z_8-equivariant planar vector field of degree 7*, J. Dyn. Differ. Equ. **16** (2004), 1123–1139.

[23] Y. Liu, J. Li, *Complete study on a bi-center problem for the Z_2-equivariant cubic vector fields*, Acta Math. Sinica **27** (2011), 1379–1394.

[24] Y. Liu, W. Huang, *A cubic system with twelve small amplitude limit cycles*, Bull. Sci. Math. **129** (2005), 83–98.

[25] J. Llibre, R.D.S. Oliveira, *Phase portraits of quadratic polynomial vector fields having a rational first integral of degree 3*, Nonlinear Anal. **70** (2009), 3549–3560.

[26] M.A. Lyapunov, *Problème général de la stabilité du mouvement*, Ann. Math. Stud., vol. **17**, Princeton University Press, 1947.

[27] K.E. Malkin, *Criteria for the center for a certain differential equation*, Volz. Mat. Sb. Vyp. **2** (1964), 87–91 (in Russian).

[28] Y. Martínez, C. Vidal, *Classification of global phase portraits and bifurcation diagrams of Hamiltonian systems with rational potential*, J. Differential Equations **261** (2016), 5923–5948.

[29] D. A. Neumann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.

[30] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, Oeuvreus de Henri Poincaré, vol. **1**, Gauthier-Villars, Paris, (1951), 3–84 and 95–114.

[31] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, J. Math. **37** (1881), 375–422.

[32] V. Romanovski, W. Fernandes, R. Oliveira, *Bi-center problem for some classes of Z_2-equivariant systems*, Comput. Appl. Math. **320** (2017), 61–75.

[33] N. Vulpe, *Affine-invariant conditions for topological distinction of quadratic systems in the presence of a center*, Differ. Equ. **19** (1983), 371–379 (in Russian).

[34] N. Vulpe, K.S. Sibirskii, *Centro-affine invariant conditions for the existence of a center of a differential system with cubic nonlinearities*, Dokl. Akad. Nauk. SSSR **301** (1988), 1297–1301 (in Russian); translation in: Soviet Math. Dokl. **38** (1989), 198–201.

[35] L. Yang, *Recent advances on determining the number of real roots of parametric polynomials*, J. Symbolic Comput. **28** (1999), 225–242.

[36] P. Yu, M. Han, *Twelve limit cycles in a cubic case of the 16th Hilbert problem*, Int. J. Bifurcation Chaos **7** (2005), 2191–2205.

[37] P. Yu, M. Han, *Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vector fields*, Chaos Solitons Fractals **24** (2005), 329–348.

[38] H. Zoladek, *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), 223–273.

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