GLOBAL DYNAMICS OF A SD OSCILLATOR

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ABSTRACT. In this paper we derive the global bifurcation diagrams of a SD oscillator which exhibits both smooth and discontinuous dynamics depending on the value of a parameter a. We research all possible bifurcations of this system, including Pitchfork bifurcation, degenerate Hopf bifurcation, Homoclinic bifurcation, Double limit cycle bifurcation, Bautin bifurcation and Bogdanov-Takens bifurcation. Besides we prove that the system has at most five limit cycles. At last, we give all numerical phase portraits to illustrate our results.

1. INTRODUCTION AND MAIN RESULTS

In recent years SD (Smooth and Discontinuous, for short) oscillator was proposed and investigated for studying the transition from smooth to discontinuous dynamics, see for instance [2, 3, 4, 5, 16]. In those papers it is proposed the elastic beam model

$$(1) \qquad \ddot{x} + \xi(b + x^2)\dot{x} + x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) = 0$$

for studying this transition, where $a \geq 0$, b and ξ can take arbitrary real values. More precisely, the smooth dynamics appears when $a > 0$, while the discontinuous dynamic behavior occurs at $a = 0$. The global dynamics was completely studied in [5] when $a = 0$, and in [16] when $|a - 1| < \varepsilon$, $|\xi| < \varepsilon$ and ε is sufficiently small.

Clearly system (1) can be rewritten as the 2-dimensional differential system

$$(2) \qquad \begin{aligned} \dot{x} &= y - \hat{\xi}(\hat{b}x + x^3) =: y - F(x), \\ \dot{y} &= -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) =: -g(x), \end{aligned}$$

where $\hat{\xi} = \xi/3$, $\hat{b} = 3b$, and for simplicity in what follows we still denote $\hat{\xi}$ and $\hat{b}$ by ξ and b respectively. Note that system (2) is invariant by the transformation $(y, t, \xi) \to (-y, -t, -\xi)$. Therefore, we only need to consider the set of parameters

$$\mathcal{G} := \{(a, b, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+\},$$

where $\mathbb{R}^+ = [0, +\infty)$.

In this paper we shall describe the dynamics of system (2). Thus, the following theorem is our main result.

Theorem 1. *System (2) has three equilibria $E_L = (-\sqrt{1-a^2}, -(1-a^2+b)\xi\sqrt{1-a^2})$, $E_0 = (0, 0)$ and $E_R = (\sqrt{1-a^2}, (1-a^2+b)\xi\sqrt{1-a^2})$ if $a < 1$, and only E_0 if $a \geq 1$. The global bifurcation diagram of system (2) consists of the following bifurcation surfaces:*
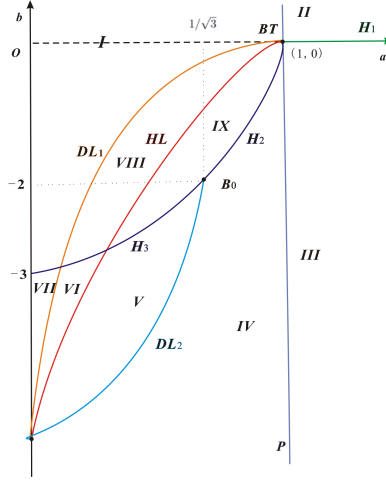

FIGURE 1. Global qualitative bifurcation diagram of system (2) on the half-plane $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ with $\xi = \xi^* > 0$ small.

(i) *Pitchfork surface* $P := \{(a, b, \xi) \in \mathcal{G} : a = 1\}$;

(ii) *Hopf surfaces*

$$H_1 := \{(a, b, \xi) \in \mathcal{G} : a > 1, b = 0\},$$
$$H_2 := \{(a, b, \xi) \in \mathcal{G} : b = 3a^2 - 3, \ 1/\sqrt{3} < a < 1\} \text{ and}$$
$$H_3 := \{(a, b, \xi) \in \mathcal{G} : b = 3a^2 - 3, \ 0 < a < 1/\sqrt{3}\};$$

(iii) *Homoclinic surface* $HL := \{(a, b, \xi) \in \mathcal{G} : b = \varphi(a, \xi), \ 0 < a < 1\}$;

(iv) *Double limit cycle surfaces*

$$DL_1 := \{(a, b, \xi) \in \mathcal{G} : b = \phi_1(a, \xi), \ 0 < a < 1\} \text{ and}$$
$$DL_2 := \{(a, b, \xi) \in \mathcal{G} : b = \phi_2(a, \xi), \ 0 < a < 1/\sqrt{3}\};$$

(v) *Codimension* 2 *Bogdanov-Takens bifurcation with symmetry curve:*

$$BT := \{(a, b, \xi) \in \mathcal{G} : a = 1, \ b = 0\};$$

(vi) *Bautin bifurcation curve:* $B_0 := \{(a, b, \xi) \in \mathcal{G} : a = 1/\sqrt{3}, \ b = -2\}$;

where

$$\varphi(a, \xi) < \phi_1(a, \xi) < a^2 - 1, \qquad \varphi(1, \xi) = \phi_1(1, \xi) = 0,$$
$$\phi_2(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}, \qquad \phi_2(1/\sqrt{3}, \xi) = -2, \text{ and}$$
$$\varphi(0, \xi) = \phi_1(0, \xi) = \phi_2(0, \xi),$$

as shown in Figure 1. Consequently, the complete classification of the phase portraits of system (2) *is given in Figure 2, where*

   I $\{(a, b, \xi) \in \mathcal{G} : b > \max\{3 - 3a^2, \ \phi_1(a, \xi)\}, \ 0 < a < 1\}$,

  II $\{(a, b, \xi) \in \mathcal{G} : b > 0, \ a > 1\}$,

 III $\{(a, b, \xi) \in \mathcal{G} : b < 0, \ a > 1\}$,

 IV $\{(a, b, \xi) \in \mathcal{G} : b < \min\{3 - 3a^2, \ \phi_2(a, \xi)\}, \ 0 < a < 1\}$,

  V $\{(a, b, \xi) \in \mathcal{G} : \phi_2(a, \xi) < b < \min\{3 - 3a^2, \varphi(a, \xi)\}, \ 0 < a < 1\}$,

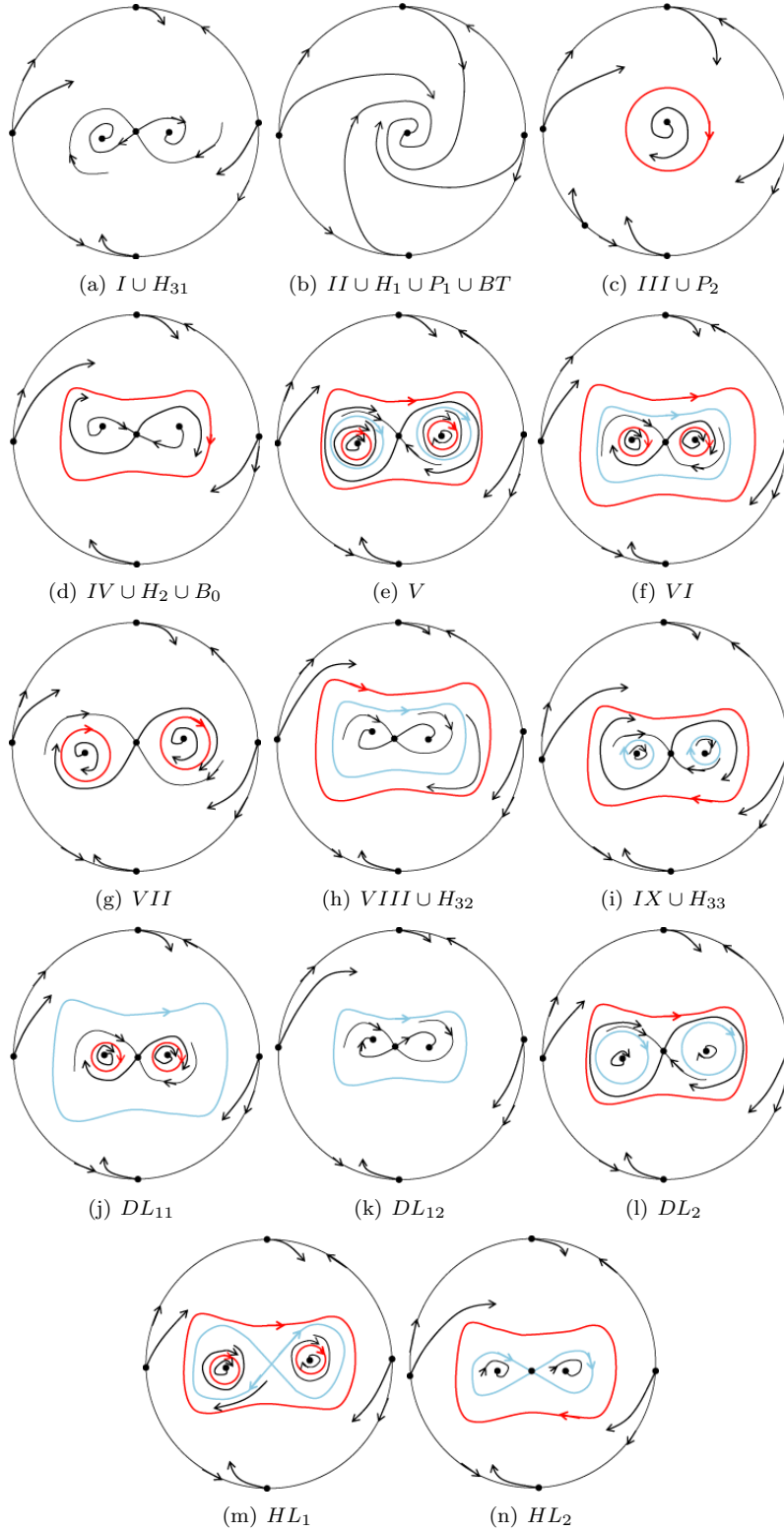 VI $\{(a, b, \xi) \in \mathcal{G} : \varphi(a, \xi) < b < \min\{3 - 3a^2, \phi_1(a, \xi)\}, \ 0 < a < 1\}$,

(a) $I \cup H_{31}$ (b) $II \cup H_1 \cup P_1 \cup BT$ (c) $III \cup P_2$

(d) $IV \cup H_2 \cup B_0$ (e) V (f) VI

(g) VII (h) $VIII \cup H_{32}$ (i) $IX \cup H_{33}$

(j) DL_{11} (k) DL_{12} (l) DL_2

(m) HL_1 (n) HL_2

FIGURE 2. Phase portraits of system (2) on the Poincaré disc.

$\text{VII} \ \{(a,b,\xi) \in \mathcal{G} : \phi_1(a,\xi) < b < 3 - 3a^2, \ 0 < a < 1\},$

$\text{VIII} \ \{(a,b,\xi) \in \mathcal{G} : \max\{3 - 3a^2, \varphi(a,\xi)\} < b < \phi_1(a,\xi), \ 0 < a < 1\},$

$\text{IX} \ \{(a,b,\xi) \in \mathcal{G} : 3 - 3a^2 < b < \varphi(a,\xi), \ 0 < a < 1\},$

$HL_1 \ \{(a,b,\xi) \in \mathcal{G} : b = \varphi(a,\xi) < 3a^2 - 3, \ 0 < a < 1\},$

$HL_2 \ \{(a,b,\xi) \in \mathcal{G} : b = \varphi(a,\xi) \geq 3a^2 - 3, \ 0 < a < 1\},$

$DL_{11} \ \{(a,b,\xi) \in \mathcal{G} : b = \phi_1(a,\xi) < 3a^2 - 3, \ 0 < a < 1\},$

$DL_{12} \ \{(a,b,\xi) \in \mathcal{G} : b = \phi_1(a,\xi) \geq 3a^2 - 3, \ 0 < a < 1\},$

$P_1 \ \{(a,b,\xi) \in \mathcal{G} : b > 0, \ a = 1\},$

$P_2 \ \{(a,b,\xi) \in \mathcal{G} : b < 0, \ a = 1\},$

$H_{31} \ \{(a,b,\xi) \in \mathcal{G} : b = 3a^2 - 3 > \phi_1(a,\xi), \ 0 < a < 1/\sqrt{3}\},$

$H_{32} \ \{(a,b,\xi) \in \mathcal{G} : \varphi(a,\xi) < b = 3a^2 - 3 < \phi_1(a,\xi), \ 0 < a < 1/\sqrt{3}\},$

$H_{33} \ \{(a,b,\xi) \in \mathcal{G} : \phi_2(a,\xi) < b = 3a^2 - 3 < \varphi(a,\xi), \ 0 < a < 1/\sqrt{3}\}.$

Moreover, all results about limit cycles and homoclinic loops are listed in Table 1.

Subsets of $\mathcal{G}$	large limit cycles surrounding all E_R, E_0, E_L	small limit cycles only surrounding $E_0 \ E_L, \ E_R$ respectively	homoclinic loops
$I, II, H_1,$ H_{31}, P_1, BT	0	0; 0; 0	0
III, P_2	0	1 stable; 0; 0	0
IV, H_2, B_0	1 stable	0; 0; 0	0
V	1 stable	0; 2 the inner one is stable, the outer one is unstable; 2 the inner one is stable, the outer one is unstable	0
VI	2 the inner one is unstable, the outer one is stable;	0; 1 stable ; 1 stable	0
VII	0	0; 1 stable; 1 stable	0
$VIII, H_{32}$	2 the inner one is unstable, the outer one is stable;	0; 0 ; 0	0
IX, H_{33}	1 stable	0; 1 unstable; 1 unstable	0
DL_{11}	1 semistable	0; 1 stable; 1 stable	0
DL_{12}	1 semistable	0; 0; 0	0
DL_2	1 stable	0; 1 semistable; 1 semistable	0
HL_1	1 stable	0; 1 stable; 1 stable	figure-eight loop, unstable
HL_2	1 stable	0; 0; 0	figure-eight loop, unstable

TABLE 1. Limit cycles and homoclinic loops of system (2).

In this paper we call *large limit cycles* to the ones surrounding all three equilibria, and *small limit cycles* the ones surrounding a single equilibrium. For the notions and definitions which appear in the statement of Theorem 1 see its proof.

The paper is organized as follows. In section 2 we analyze the local bifurcations, namely pitchfork bifurcation, Hopf bifurcation, Bautin bifurcation and codimension 2 Bogdanov-Takens bifurcation with symmetry. In section 3 we estimate the number of limit cycles in difference parameter regions and curves. In section 4 we study the global bifurcations, namely the different kinds of homoclinic connections and the

double limit cycles. In section 5 we give the numerical phase portraits in difference parameter regions.

2. Local bifurcations

Computing the Jacobian matrix at equilibrium E_0, we have

$$J_0 := \begin{pmatrix} -b\xi & 1 \\ \frac{1}{a} - 1 & 0 \end{pmatrix}.$$

Then, at E_0 the determinant $det(J_0) = 1 - 1/a$ and the trace $tr(J_0) = -b\xi$, implying that E_0 is a saddle if $a < 1$, stable focus or node if $a > 1$ and $b > 0$, and unstable focus or node if $a > 1$ and $b < 0$. When $a = 1$, equilibrium E_0 is a stable node if $b > 0$, and unstable node if $b < 0$ by applying [8, Theorem 2.19]. By the symmetry of the vector field (2), equilibrium E_L is of the same type as E_R. The Jacobian matrix at E_R is

$$J_R := \begin{pmatrix} -\xi(b + 3 - 3a^2) & 1 \\ -(1 - a^2) & 0 \end{pmatrix}.$$

Calculating $det(J_R) = 1 - a^2$ and $tr(J_R) = -\xi(b + 3 - 3a^2)$. Hence E_R is a stable focus or node if $a < 1$ and $b + 3 - 3a^2 > 0$, and unstable focus or node if $a < 1$ and $b + 3 - 3a^2 < 0$.

2.1. **Pitchfork bifurcation.** From the expressions of the equilibria E_L, E_0 and E_R it follows immediately that a pitchfork bifurcation occurs at the origin of coordinates when $a = 1$; i.e. for $a \geq 1$ we have a unique antisaddle, while for $0 < a < 1$ from the previous antisaddle it bifurcates at $a = 1$ a saddle and two antisaddles. For more details on this kind of bifurcation see [7, 10].

2.2. **Hopf bifurcations.** There are two kinds of Hopf bifurcations, one at the equilibrium E_0, and the other at the equilibria E_L and E_R, which is essentially the same bifurcation in both, because due to the invariance of system (2) with respect to the symmetry $(x, y) \mapsto (-x, -y)$, what occurs at the equilibrium point E_L occurs to its symmetric E_R.

2.2.1 **Hopf bifurcation at** E_0. The next result characterizes the Hopf bifurcation at the equilibrium point E_0, it is proved using the averaging theory in this way we also can estimated the shape of the limit cycle bifurcating from E_0, and we avoid the computation of the Liapunov constant.

Proposition 2. *The following statements hold for the differential system* (2).

(a) *If* $a > 1$, $b = 0$ *and* $\xi > 0$, *then a Hopf bifurcation takes place at the equilibrium point located at the origin of coordinates, and the limit cycle* γ *bifurcating from this equilibrium exists for* $b < 0$ *sufficiently small.*

(b) *For* $\varepsilon > 0$ *sufficiently small if* $b = -\beta\varepsilon^2 < 0$, *then the limit cycle* γ *passes through the point*

$$\left(2\sqrt{\frac{\beta}{3}}\varepsilon + O(\varepsilon^3), 0 \right).$$

Moreover, this limit cycle is stable.

Proof. Assume that $a > 1$, $\xi > 0$ and $b = -\beta\varepsilon^2 < 0$ where ε is a sufficiently small parameter. Writing the differential system (2) in polar coordinates we get

(3)
$$\dot{r} = \frac{r\cos\theta\sin\theta}{\sqrt{a^2 + r^2\cos^2\theta}} - \xi r^3 \cos^4\theta + \varepsilon^2 \xi\beta\, r\cos^2\theta,$$

$$\dot{\theta} = -1 + \frac{\cos^2\theta}{\sqrt{a^2 + r^2\cos^2\theta}} + \xi r^2 \sin\theta\cos^3\theta - \varepsilon^2 \xi\beta\cos\theta\sin\theta.$$

Since we want to study the Hopf bifurcation at the origin of coordinates we blow up the origin doing the scaling $r = \varepsilon R$, then differential system (3) taking as new independent variable the θ becomes

(4)
$$\frac{dR}{d\theta} = \frac{R\sin\theta\cos\theta}{\cos^2\theta - a} +$$
$$\varepsilon^2 \frac{R\left(2\xi a^2(a-1)(R^2 - 2\beta R^2\cos(2\theta)) + R^2\sin(2\theta)\right)\cos^2\theta}{4a(\cos^2\theta - a)^2} + O(\varepsilon^4),$$

In order to apply the averaging theory described in the appendix we need that the differential equation (4) starts at least with order ε. So we do the change of variables $R \mapsto \rho$ defined by

$$R = \sqrt{\frac{2(1-a)}{1 - 2a + \cos(2\theta)}}\,\rho.$$

Then differential equation (4) in the new variable ρ writes

(5)
$$\frac{d\rho}{d\theta} = -\varepsilon^2 \frac{2(a-1)\rho\cos^2\theta}{a(1 - 2a + \cos(2\theta))^3} \left(2\xi a^2\left((a-1)\rho^2 - 2a\beta + \beta\right) +\right.$$
$$\left. 2\xi a^2\left((a-1)\rho^2 + \beta\right)\cos(2\theta) + \rho^2\sin(2\theta)\right) + O(\varepsilon^4) +$$
$$= \varepsilon^2 F_1(\theta, \rho) + O(\varepsilon^4).$$

Differential equation (5) is written into the normal form (37) for applying the averaging theory summarized in the appendix, using the notation of the appendix we only need to take $n = 1$, $\mathbf{x} = \rho$, $t = \theta$, $\mu = \varepsilon^2$, $F_1(t, \mathbf{x}) = F_1(\theta, \rho)$ and $T = 2\pi$, all the necessary assumptions for applying the averaging theory described in the appendix are satisfied. Then we compute

$$f_1(\rho) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, \rho)d\theta = \frac{\xi}{8}\sqrt{\frac{a}{a-1}}\rho(3\rho^2 - 4\beta).$$

Since $\beta > 0$ the averaged function $f_1(\rho)$ has a unique positive zero, $\rho = 2\sqrt{\beta/3}$ which satisfies the condition

(6)
$$D_\rho f_1\left(2\sqrt{\frac{\beta}{3}}\right) = \xi\sqrt{\frac{a}{a-1}} \neq 0.$$

In fact this last expression is positive because $a > 1$ and $\xi > 0$, and consequently, by the results described in the appendix the differential equation (5) has a periodic solution $\rho(\theta, \varepsilon)$ satisfying that

(7)
$$\rho(0,0) = 2\sqrt{\frac{\beta}{3}} + O(\varepsilon^2).$$

Moreover, this periodic solution $\rho(\theta, \varepsilon)$ is unstable because the derivative (6) is positive.

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium at the origin of coordinates of the differential system (2). Thus, the periodic solution $\rho(\theta, \varepsilon)$ satisfying the initial condition (7) in the variables of the differential system (4) becomes the periodic solution

$$R(\theta, \varepsilon) = \sqrt{\frac{2(1-a)}{1 - 2a + \cos(2\theta)}} \, \rho(\theta, \varepsilon),$$

satisfying

$$R(0,0) = 2\sqrt{\frac{\beta}{3}} + O(\varepsilon^2).$$

This periodic solution in the differential system (3) becomes $(r(t,\varepsilon), \theta(t,\varepsilon))$ with

$$r(t,\varepsilon) = \sqrt{\frac{2(1-a)}{1 - 2a + \cos(2\theta(t,\varepsilon))}} \, \rho(\theta(t,\varepsilon), \varepsilon)\varepsilon + O(\varepsilon^3),$$

and it pass through the point

$$(8) \qquad\qquad \left(2\sqrt{\frac{\beta}{3}}\varepsilon + O(\varepsilon^3), 0 \right)$$

in the coordinates (r, θ). Finally, this periodic solution in the coordinates of system (2) is the periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ given by

$$\varepsilon\sqrt{\frac{2(1-a)}{1 - 2a + \cos(2\theta(t,\varepsilon))}} \, \rho(\theta(t,\varepsilon), \varepsilon)\big(\cos(\theta(t,\varepsilon)), \sin(\theta(t,\varepsilon)) \big) + O(\varepsilon^3),$$

passing through the point (8) now in coordinates (x, y). Therefore, when $\varepsilon \mapsto 0$ such periodic solution tends to the origin, so it is a periodic solution of a Hopf bifurcation.

We remark that the periodic solution $\rho(\theta, \varepsilon)$ was an unstable limit cycle, but due to the fact that $\dot\theta$ is negative in a neighborhood of the origin, when we pass the unstable limit cycle $R(\theta, \varepsilon)$ to the periodic solution $(r(t,\varepsilon), \theta(t,\varepsilon))$ it changes to a stable limit cycle. $\qquad\square$

In short, Proposition 2 shows the existence of the Hopf bifurcation surface H_1.

2.2.2 Hopf bifurcation at E_L and E_R.

Proposition 3. *The following statements hold for the differential system (2).*

(a) *If $0 < a < 1$, $a \neq 1/\sqrt{3}$, $b = 3(a^2 - 1)$ and $\xi > 0$, then one limit cycle bifurcates from each one of the equilibria E_L and E_R.*

(b) *For $\varepsilon > 0$ sufficiently small if $b = 3a^2 - 3 + \beta\varepsilon^2$, then those two limit cycles exist if $\beta(1 - 3a^2) < 0$, and they pass through the points*

$$\pm \left(\sqrt{1 - a^2} + 2\sqrt{\frac{\beta(1 - a^2)}{3(3a^2 - 1)}} \, \varepsilon + O(\varepsilon^2), \xi\sqrt{1 - a^2}(2a^2 - 2 + \beta\varepsilon^2) \right).$$

Moreover, these limit cycles are stable if $\beta < 0$, and unstable if $\beta > 0$.

Proof. Assume $0 < a < 1$, $a \neq 1/\sqrt{3}$, $\xi > 0$ and $b = 3(a^2 - 1) + \beta \varepsilon^2$, where ε is a sufficiently small parameter. We shall prove the proposition studying the Hopf bifurcation at the equilibrium point E_R.

We translate the equilibrium point E_R to the origin of coordinates doing the change

$$(x, y) = (X + \sqrt{1 - a^2}, Y + \xi\sqrt{1 - a^2}(2a^2 - 2 + \beta \varepsilon^2)),$$

and system (2) is transformed into

(9)
$$\dot{X} = -\varepsilon^2 \xi \beta X + Y - 3\xi\sqrt{1 - a^2}X^2 - \xi X^3,$$

$$\dot{Y} = \left(-X - \sqrt{1 - a^2}\right)\left(1 - \frac{1}{\sqrt{X^2 + 2\sqrt{1 - a^2}X + 1}}\right).$$

Writing the differential system (9) in polar coordinates we get
(10)
$$\dot{r} = -r\cos\theta\left(\xi r^2 \cos^3\theta + 3\xi\sqrt{1 - a^2}r\cos^2\theta - \sin\theta\right) - \left(r\cos\theta + \sqrt{1 - a^2}\right)$$

$$\cdot \left(1 - \frac{1}{\sqrt{r^2\cos^2\theta + 2\sqrt{1 - a^2}r\cos\theta + 1}}\right)\sin\theta - \varepsilon^2 \xi \beta r \cos^2\theta,$$

$$\dot{\theta} = -1 + \xi\left(r^2\cos^2\theta + 3\sqrt{1 - a^2}r\cos\theta\right)\sin\theta\cos\theta +$$

$$\frac{\left(r\cos\theta + \sqrt{1 - a^2}\right)\cos\theta}{r\sqrt{r^2\cos^2\theta + 2\sqrt{1 - a^2}r\cos\theta + 1}} - \frac{\sqrt{1 - a^2}\cos\theta}{r}.$$

Again since we want to study the Hopf bifurcation now at the origin of coordinates we blow up the origin doing the scaling $r = \varepsilon R$, then differential system (10) taking as new independent variable the θ writes

$$\frac{dR}{d\theta} = \frac{a^2 R\cos\theta\sin\theta}{a^2\cos^2\theta - 1} - \varepsilon\frac{3\sqrt{1 - a^2}R^2\cos^2\theta\left(2\xi\left(a^2 - 1\right)\cos\theta - a^2\sin\theta\right)}{2\left(a^2\cos^2\theta - 1\right)^2}$$

$$+ \varepsilon^2\frac{R\cos^2\theta}{32\left(a^2\cos^2\theta - 1\right)^3}\Big(54\xi a^6 R^2 + 2a^6 R^2\sin(2\theta) + a^6 R^2\sin(4\theta)$$

$$- 102\xi a^4 R^2 - 16\xi\beta a^4 - 72\xi^2 a^4 R^2\sin(2\theta) - 38a^4 R^2\sin(2\theta)$$

(11)
$$- 36\xi^2 a^4 R^2\sin(4\theta) + a^4 R^2\sin(4\theta) + 64\xi a^2 R^2 + 48\xi\beta a^2$$

$$+ 2\xi a^2\left(9a^4 - 29a^2 + 20\right)R^2\cos(4\theta) + 144\xi^2 a^2 R^2\sin(2\theta)$$

$$+ 32a^2 R^2\sin(2\theta) + 72\xi^2 a^2 R^2\sin(4\theta) - 16\xi R^2 - 32\xi\beta$$

$$+ 8\xi\left(a^2 - 1\right)\left(9R^2 a^4 - a^2\left(11R^2 + 2\beta\right) + 2R^2\right)\cos(2\theta)$$

$$- 72\xi^2 R^2\sin(2\theta) - 36\xi^2 R^2\sin(4\theta)\Big) + O(\varepsilon^3),$$

Again for applying the averaging theory of the appendix we need that the differential equation (11) starts at least with order ε. Hence we do the change of variables $R \mapsto \rho$ defined by

$$R = \sqrt{\frac{2(1 - a^2)}{2 - a^2 + a^2\cos(2\theta)}}\,\rho.$$

Then differential equation (11) in the new variable ρ writes

$$(12) \qquad \frac{d\rho}{d\theta} = \varepsilon F_1(\theta, \rho) + \varepsilon^2 F_2(\theta, \rho) + O(\varepsilon^3),$$

where

$$F_1(\theta, \rho) = \frac{3\left(a^2 - 1\right) \rho^2 \cos^2 \theta \left(2\xi\left(a^2 - 1\right) \cos\theta - a^2 \sin\theta\right)}{\sqrt{2}\left(a^2 \cos^2 \theta - 1\right)^2 \sqrt{2 - a^2 - a^2 \cos(2\theta)}},$$

$$\begin{aligned} F_2(\theta, \rho) = \quad & \frac{\left(a^2 - 1\right) \rho \cos^2 \theta}{2\left(\cos(2\theta)a^2 + a^2 - 2\right)^4}\Big(54e\rho^2 a^6 + 2\rho^2 \sin(2\theta)a^6 + \rho^2 \sin(4\theta)a^6 \\ & -102e\rho^2 a^4 - 12e\beta a^4 - 72e^2 \rho^2 \sin(2\theta)a^4 - 38\rho^2 \sin(2\theta)a^4 \\ & -36e^2 \rho^2 \sin(4\theta)a^4 + \rho^2 \sin(4\theta)a^4 + 64e\rho^2 a^2 + 32e\beta a^2 \\ & +2e\left(9\rho^2 a^4 - \left(29\rho^2 + 2\beta\right)a^2 + 20\rho^2\right)\cos(4\theta)a^2 \\ & +144e^2\rho^2 \sin(2\theta)a^2 + 32\rho^2 \sin(2\theta)a^2 + 72e^2\rho^2 \sin(4\theta)a^2 \\ & -16e\rho^2 - 32e\beta + 8e\big(9\rho^2 a^6 - 2\left(10\rho^2 + \beta\right)a^4 \\ & +\left(13\rho^2 + 4\beta\right)a^2 - 2\rho^2\big)\cos(2\theta) - 72e^2\rho^2 \sin(2\theta) \\ & -36e^2\rho^2 \sin(4\theta)\Big). \end{aligned}$$

Differential equation (12) is already into the normal form (37) for applying the averaging theory of the appendix. Again using the notation of the appendix we take $n = 1$, $\mathbf{x} = \rho$, $t = \theta$, $\mu = \varepsilon$, $F_1(t, \mathbf{x}) = F_1(\theta, \rho)$ and $T = 2\pi$, and all the necessary hypotheses for applying the averaging theory of the appendix hold. Then we compute

$$f_1(\rho) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, \rho)d\theta \equiv 0.$$

Since the first averaged function $f_1(\rho)$ is identically zero, we must compute the second one $f_2(\rho)$. We start calculating

$$\int_0^\theta F_1(\theta, \rho)ds = \frac{\rho^2 N(\theta)}{2\sqrt{2 - 2a^2}\left(2 - a^2 - a^2 \cos(2\theta)\right)^{3/2}},$$

where

$$\begin{aligned} N(\theta) = \quad & 3a^2(1 - a^2)^{3/2}\cos\theta - a^2\sqrt{2}(2 - a^2 - a^2 \cos(2\theta))^{3/2} \\ & +\sqrt{1 - a^2}\Big((a^2 - a^4)\cos(3\theta) + 2\xi\big(3(a^2 - 3)\sin\theta \\ & +(3a^2 - 1)\sin(3\theta)\big)\Big). \end{aligned}$$

Then the second averaged function

$$\begin{aligned} f_2(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\left[D_\rho F_1(\theta, \rho)\int_0^\theta F_1(s, \rho)ds + F_2(\theta, \rho)\right]d\theta \\ &= \frac{\xi\rho\left(3\left(1 - 3a^2\right)\rho^2 + 4\beta\right)}{8\sqrt{1 - a^2}} \end{aligned}$$

has a unique positive zero, $\rho = 2\sqrt{\beta/(3(3a^2-1))}$, recall that $a \neq 1/\sqrt{3}$. This zero satisfies the condition

$$(13) \qquad D_\rho f_2 \left(2\sqrt{\frac{\beta}{3(3a^2-1)}} \right) = -\frac{\xi\beta}{\sqrt{1-a^2}} \neq 0,$$

by assumptions. Consequently, by the results described in the appendix the differential equation (12) has a periodic solution $\rho(\theta, \varepsilon)$ satisfying that

$$(14) \qquad \rho(0,0) = 2\sqrt{\frac{\beta}{3(3a^2-1)}} + O(\varepsilon).$$

Moreover, from (13) this periodic solution $\rho(\theta, \varepsilon)$ is unstable if $\beta < 0$, and stable if $\beta > 0$.

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium E_R of the differential system (2). Thus, the periodic solution $\rho(\theta, \varepsilon)$ satisfying the initial condition (14) in the variables of the differential system (11) becomes the periodic solution

$$R(\theta, \varepsilon) = \sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta)}} \; \rho(\theta, \varepsilon),$$

satisfying

$$R(0,0) = 2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}} + O(\varepsilon).$$

This periodic solution in the differential system (10) becomes $(r(t,\varepsilon), \theta(t,\varepsilon))$ with

$$r(t,\varepsilon) = \sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta(t,\varepsilon))}} \; \rho(\theta(t,\varepsilon), \varepsilon)\varepsilon + O(\varepsilon^2),$$

and it pass through the point

$$(15) \qquad \left(2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}} \, \varepsilon + O(\varepsilon^2), 0 \right)$$

in the coordinates (r, θ). This periodic solution in the coordinates of system (9) is the periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ given by

$$\varepsilon\sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta(t,\varepsilon))}} \; \rho(\theta(t,\varepsilon),\varepsilon)\big(\cos(\theta(t,\varepsilon)), \sin(\theta(t,\varepsilon)) \big) + O(\varepsilon^2),$$

passing through the point (15) now in coordinates (X, Y). Finally, we get the periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ given by

$$(x(t,\varepsilon), y(t,\varepsilon)) = (X(t,\varepsilon) + \sqrt{1-a^2}, Y(t,\varepsilon) + \xi\sqrt{1-a^2}(2a^2-2+\beta\varepsilon^2)),$$

and passing through the point

$$\left(\sqrt{1-a^2} + 2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}} \, \varepsilon + O(\varepsilon^2), \xi\sqrt{1-a^2}(2a^2-2+\beta\varepsilon^2) \right)$$

in coordinates (x, y). So, when $\varepsilon \mapsto 0$ such periodic solution tends to the equilibrium E_R, so it is a periodic solution of a Hopf bifurcation.

Again we note that the limit cycle $\rho(\theta, \varepsilon)$ was unstable if $\beta < 0$, and stable if $\beta > 0$, but due to the fact that $\dot{\theta}$ is negative in a neighborhood of the origin, when we pass the limit cycle $R(\theta, \varepsilon)$ to the cycle $(r(t, \varepsilon), \theta(t, \varepsilon))$ it changes its type of stability. $\qquad\square$

In particular Proposition 3 shows the existence of the Hopf bifurcation surfaces H_2 and H_3.

2.2.3 The Bautin bifurcation curve B_0. The standard or classical Hopf bifurcation in a 2-dimensional differential system, i.e. that a limit cycle bifurcates from an equilibrium point, takes place in an equilibrium point with purely imaginary eigenvalues which is not a center because the first Liapunov constant at that equilibrium is not zero. These are the Hopf bifurcations studied in subsection 2.2.2. But when the first Liapunov constant is zero, also can bifurcate a limit cycle of the equilibrium point if the second Liapunov constant is not zero, such more degenerate Hopf bifurcation is called for some authors a *Bautin bifurcation*. See for more details about these Hopf bifurcations [11, Chapter 8].

When $b = 3(a^2 - 1)$, with the change of variables $x_1 = x - \sqrt{1 - a^2}$, $y_1 = y/\sqrt{1 - a^2} - \xi(b + 1 - a^2)$ and $\tau = \sqrt{1 - a^2}\,t$ in a small neighborhood of E_R, system (2) can be written as

$$
\text{(16)} \quad
\begin{aligned}
\frac{dx}{dt} &= y - 3\xi x^2 - \frac{\xi}{\sqrt{1 - a^2}} x^3 =: y + \hat{f}(x), \\
\frac{dy}{dt} &= -x - \frac{3a^2 x^2}{2\sqrt{1 - a^2}} + \frac{(4a^2 - 5a^4)x^3}{2 - 2a^2} + O(x^4) =: -x + \hat{h}(x),
\end{aligned}
$$

where, for simplicity, we still use the variables (x, y, t) instead of the new ones (x_1, y_1, τ). From [10, p. 156] we compute the first Liapunov constant at the origin of system (16) and we get $\hat{g}_1 = 3\xi(3a^2 - 1)/(8\sqrt{1 - a^2})$. From the expressions of $\hat{g}_1$, we can confirm that the classical Hopf bifurcation happens when $a \neq 1/\sqrt{3}$ and $b = 3a^2 - 3$. Clearly, when $a = 1/\sqrt{3}$ and $b = -2$ we have $\hat{g}_1 = 0$. By [11, Chapter 8], we can obtain that the second Liapunov constant $\hat{g}_2 = 5\sqrt{6}\,\xi/32 > 0$ at those values of a and b.

In short, system (2) exhibits a Bautin bifurcation at $a = 1/\sqrt{3}$ and $b = -2$ by [11, Chapter 8], i.e. at the intersection point of the surfaces H_2, H_3 and DL_2, in particular $\phi_2(1/\sqrt{3}, \xi) = -2$.

2.3. Codimension 2 Bogdanov-Takens bifurcation with symmetry. From [16] a codimension 2 Bogdanov-Takens bifurcation with symmetry happens in system (2) in a neighborhood of the curve $a = 1$ and $b = 0$. So, in a neigborhood of the intersection point of P, H_1, H_2, HL and DL_1 of the bifurcation diagram of Figure 1, i.e. $\varphi(1, \xi) = \phi_1(1, \xi) = 0$.

2.4. The dynamics near infinity. In this subsection we will discuss the qualitative properties of the equilibria at infinity, which describe the behavior of the orbits of system (2) when x and y are sufficiently large.

Proposition 4. *As shown in Figure 3 the differential system* (2) *with $\xi > 0$ has four equilibria at infinity $I_A^{\pm}, I_B^{\pm}$, where $I_A^{\pm}$ are the two endpoints of the x-axis, and $I_B^{\pm}$ are the two endpoints of the y-axis. The equilibria $I_A^{\pm}$ are unstable star*
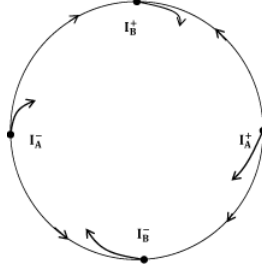
FIGURE 3. Equilibria at infinity.

nodes, and the equilibria $I_B^\pm$ are degenerate equilibria, formed by the union of two hyperbolic sectors.

Proof. Doing the Poincaré transformation $x = 1/z$, $y = u/z$, system (2) becomes

$$\frac{du}{d\tau} = \xi u + b\xi u z^2 - u^2 z^2 - z^2 + \frac{|z|^3}{\sqrt{1 + a^2 z^2}},$$

$$\frac{dz}{d\tau} = \xi z + b\xi z^3 - u z^3,$$

where $dt = z^2 d\tau$. Obviously, this system has a unique equilibrium $A : (0,0)$ on the u-axis (the infinity), where A is an unstable star node.

Doing the other Poincaré transformation $x = v/z, y = 1/z$, system (2) writes as

(17)
$$\frac{dv}{dt} = z^2 - b\xi v z^2 - \xi v^3 + v^2 z^2 - \frac{v^2 |z|^3}{\sqrt{v^2 + a^2 z^2}} := \Psi_1(v, z),$$

$$\frac{dz}{dt} = v z^3 - \frac{v z^3 |z|}{\sqrt{v^2 + a^2 z^2}} := \Psi_2(v, z),$$

where $dt = z^2 d\tau$. In this local chart we only need to study the equilibrium $B : (0,0)$ of system (17), which corresponds to two equilibria I_{B+} and I_{B-} at infinity of the system (2) at the endpoints of the positive and negative y-semiaxes, respectively. By Lemmas 1 and 3 of [14, Chapter 2] we only need to discuss the orbits along characteristic directions of system (17) at B.

Applying the polar coordinate changes $x = r\cos\theta$ and $y = r\sin\theta$, system (17) can be written
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H(\theta) + o(1)}{G(\theta) + o(1)}, \quad \text{as } r \to 0,$$
where

$$G(\theta) = -\sin^3\theta\sqrt{\cos^2\theta + a^2\sin^2\theta}, \quad H(\theta) = \sin^2\theta\cos\theta\sqrt{\cos^2\theta + a^2\sin^2\theta}.$$

Hence a necessary condition for $\theta = \theta_0$ to be an characteristic direction is $G(\theta_0) = 0$, which has exactly two roots 0 and π. Except these two directions, there are no directions along which system (17) has orbits connecting B.

Notice that the vector field (17) is symmetric with respect to the v–axis. Thus, we only need to discuss the orbits connecting the origin B of (17) in the half plane $z \geq 0$. We will construct some related open quasi–sectors to determine how many orbits of (17) connecting B in the first and the second quadrants.

Observing that system (17) has four horizontal isoclines: the v-axis, the z-axis and

$$\mathcal{H}^{\pm} := \left\{ (u, z) \in \mathbb{R}^2 : v = \pm\sqrt{1 - a^2}z, \ 0 < r < \ell, \ a < 1 \right\},$$

where $\ell > 0$ is a sufficiently small constant. Set

$$V^{\pm} := \left\{ (v, z) \in \mathbb{R}^2 : z = 0, \ \pm v > 0, \ 0 < r < \ell \right\}.$$

The possible vertical isocline is

$$\mathcal{V} := \left\{ (v, z) \in \mathbb{R}^2 : v = \xi^{-\frac{1}{3}} z^{\frac{2}{3}} + o(z^{\frac{2}{3}}), \ 0 < r < \ell \right\}.$$

Obviously, the isocline $\mathcal{V}$ is tangent to the v–axis at the origin. Set

$$\mathcal{L}^{\pm} := \left\{ (v, z) \in \mathbb{R}^2 : z = \pm \sigma v, \ 0 < r < \ell, \ a = 1 \right\},$$

where $\sigma > 0$ is a small constant. Hence, if there exist orbits of system (17) connecting B along the direction of the v–axis in the first and the second quadrants, then near the origin must lie in the sector regions $\Delta \widehat{V^+ B \mathcal{H}^+}$ or $\Delta \widehat{V^- B \mathcal{H}^-}$ if $a < 1$, and $\Delta \widehat{V^+ B \mathcal{L}^+}$ or $\Delta \widehat{V^- B \mathcal{L}^-}$ if $a = 1$. The directions of vector field of (17), i.e., the directions of arrows, and the positions of the isoclines are shown in Figure 4.
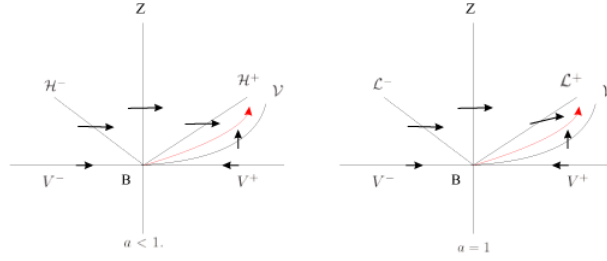

FIGURE 4. The vector field of system (17).

Firstly, we consider the case $a < 1$. We can check that $\dot{v} > 0$ and $\dot{z} > 0$ in $\Delta \widehat{\mathcal{V} B \mathcal{H}^+}$; $\dot{v} < 0$ and $\dot{z} > 0$ in $\Delta \widehat{V^+ B \mathcal{V}}$; and $\dot{v} > 0$ and $\dot{z} < 0$ in $\Delta \widehat{V^- B \mathcal{H}^-}$.

Lemma 4 in [15] guarantees that no orbits connect B in $\Delta \widehat{V^+ B \mathcal{V}}$. There are also no orbits connecting B in the interior of $\Delta \widehat{V^- B \mathcal{H}^-}$, because $\Psi_2(v, z)/\Psi_1(v, z)$ is not equal to the slopes of the curves tangent to the v–axis. On the other hand, we compute that $(\partial/\partial v)(\Psi_1(v, z)/\Psi_2(v, z)) < 0$ in the generalized normal sector $\Delta \widehat{\mathcal{V} B \mathcal{H}^+}$ of class II, i.e. $\dot{r} > 0$ in $\Delta \widehat{\mathcal{V} B \mathcal{H}^+}$ and all positive semi-orbits starting from the curves $B\mathcal{V}$ and $B\mathcal{H}^+$ go into $\Delta \widehat{\mathcal{V} B \mathcal{H}^+}$. The definition of generalized normal sectors can be seen in [15, Section 2]. Therefore, there exists a unique orbit leaving from B in $\Delta \widehat{\mathcal{V} B \mathcal{H}^+}$ by Lemma 2 and Lemma 5 in [15].

Similarly, in case $a = 1$, we can also prove that exactly one orbit connects B along the v–axis, which lies in $\Delta \widehat{\mathcal{V} B \mathcal{L}^+}$. □

3. Limit cycles

Lemma 5. *Assume that $a \geq 1$. System (2) has no limit cycles if $b \geq 0$, and a unique limit cycle if $b < 0$.*

Proof. When $b \geq 0$, since the divergence of system (2) is $f(x) = F'(x) = \xi(b + 3x^2) \geq 0$, by the Bendixson criterion (see for instance [8, Theorem 7.10]), the system (2) has no limit cycles.

When $b < 0$, the following conditions are satisfied.

(i) $g(x)$ is an odd function and $xg(x) > 0$ if $x \neq 0$;
(ii) $F(x)$ is an odd function, $F(x) < 0$ if $0 < x < \sqrt{-b}$, and $F(x) \geq 0$ if $x \geq \sqrt{-b}$;
(iii) $\int_0^\infty f(x)dx = \int_0^\infty g(x)dx = +\infty$;
(iv) $f(x)$ and $g(x)$ satisfy the Lipschitz condition in any bounded interval.

Then, by Theorem 4.1 of [18, Chapter 4], the system (2) has a unique limit cycle, which is stable.   □

Note that the phase portraits (b) and (c) of Figure 2 in Theorem 1 are obtained from Lemma 5 and from the properties of equilibria.

Since the phase portrait of system (2) is symmetric with respect to the point E_0, the small limit cycles surrounding E_L are of the same type as that surrounding E_R. Hence, in what follows we only consider the small limit cycles around E_R.

Lemma 6. *If $0 < a < 1$ and $b \geq a^2 - 1$, then system (2) has no limit cycles.*

Since the proof is similar to Lemma 4 of [5], we omit it.

Lemma 6 shows that $\phi_1(a, \xi) < a^2 - 1$ and the phase portrait (a) of Figure 2 in Theorem 1 is obtained.

Consider equation

$$(18) \qquad \frac{dz}{dy} = y - \hat{F}(z), \ 0 \leq z \leq z_0,$$

where both $\hat{F}(z)$ and $\hat{F}'(z)$ are continuous in $[0, z_0)$, and $\hat{F}(0) = 0$. Let L_J denote the integral curve of (18) passing through the point $P(z_J, \hat{F}(z_J))$ on the curve $y = \hat{F}(z)$. Also, let $y = \varphi_J(z)$ and $y = \tilde{\varphi}_J(z)$ represent the orbit segments of L_J below and above the curve $y = \hat{F}(z)$. When $0 < z < z_J$, we clearly have $\varphi_J(z) < \hat{F}(z) < \tilde{\varphi}_J(z)$ and $\varphi'_J(z) > 0 > \tilde{\varphi}'_J(z)$. Moreover, we introduce the symbol

$$(19) \qquad V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z)) = \frac{\hat{F}'(z)}{\hat{F}(z) - \varphi_J(z)} + \frac{\hat{F}'(z)}{\tilde{\varphi}_J(z) - \hat{F}(z)}.$$

Then, we have

$$(20) \qquad \int_{L_J} \hat{F}'(z)dy = \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z))dz$$

for some a_0.

Lemma 7. *[Lemma 4.5 of [18, Chapter 4]] For equation (18), suppose there is $a_0 \geq 0$ with $\hat{F}(a_0) = 0$, and $\hat{F}(z) > 0$, $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > a_0$. Then*

$$\int_{a_0}^{z_Q} V(\hat{F}(z), \varphi_Q(z), \tilde{\varphi}_Q(z))dz \leq \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z))dz, \text{ for } a_0 < z_Q < z_J.$$

Lemma 7 will be applied in the following lemma.

Lemma 8. *If $0 < a < 1$ and $b < a^2 - 1$, then system (2) has at most two large limit cycles.*
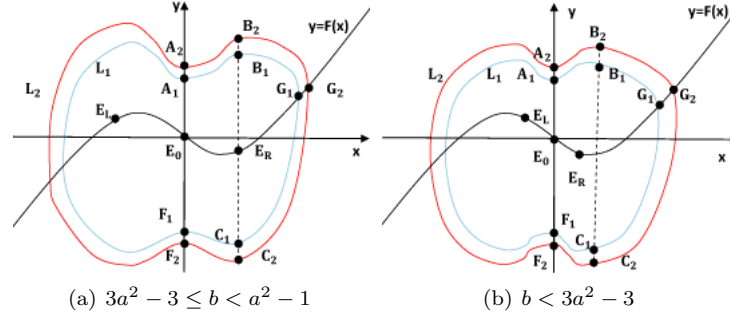

(a) $3a^2 - 3 \leq b < a^2 - 1$ (b) $b < 3a^2 - 3$

FIGURE 5. Two large limit cycles.

Proof. Assume that system (2) has at least two large limit cycles surrounding the three equilibria E_L, E_0 and E_R, and that L_1 and L_2 are the most external limit cycles, where L_2 denotes the outer one.

We first consider $3a^2 - 3 \leq b < a^2 - 1$. The corresponding phase portrait is shown in Figure 5(a). By the Bendixson criterium, each L_i has two intersection points, denoted by B_i and C_i $(i = 1, 2)$ with the straight line $x = x_0$, where x_0 is the abscissa of the equilibrium E_R, as shown in Figure 5(a). By the symmetry of the phase portrait

$$2 \int_{\widehat{A_i F_i}} f(x)dt = \oint_{L_i} f(x)dt = - \oint_{L_i} \mathrm{div}(y - F(x), -g(x))dt,$$

for $i = 1, 2$. On the arcs $\widehat{A_1 B_1}$ and $\widehat{A_2 B_2}$, let $y = y_1(x)$ and $y = y_2(x)$, respectively. In fact, for each $i = 1, 2$, we have

$$\int_{\widehat{A_i B_i}} f(x)dt = -\int_0^{x_0} \frac{F'(x)}{F(x) - y_i(x)}dx = -\int_0^{x_0} \frac{F'(x) - y_i'(x) + y_i'(x)}{F(x) - y_i(x)}dx$$

$$= -\int_0^{x_0} \frac{1}{F(x) - y_i(x)}d(F(x) - y_i(x)) - \int_0^{x_0} \frac{y_i'(x)}{F(x) - y_i(x)}dx$$

$$= -\ln\left|\frac{F(0) - y_i(x_0)}{F(0) - y_i(0)}\right| - \ln\left|\frac{F(x_0) - y_i(x_0)}{F(0) - y_i(x_0)}\right| - \int_0^{x_0} \frac{g(x)}{(F(x) - y_i(x))^2}dx$$

$$= -\ln\left|\frac{F(x_0) - y_i(x_0)}{F(0) - y_i(x_0)}\right| + \int_0^{x_0} \frac{y_i'(x)}{F(0) - y_i(x)}dx - \int_0^{x_0} \frac{g(x)}{(F(x) - y_i(x))^2}dx$$

$$= -\ln\left|\frac{y_i(x_0) - F(x_0)}{y_i(x_0)}\right| - \int_0^{x_0} \frac{F(x)g(x)}{y_i(x)(F(x) - y_i(x))^2}dx,$$

implying

$$\int_{\widehat{A_2 B_2}} F'(x)dt - \int_{\widehat{A_1 B_1}} F'(x)dt = \ln\left|\frac{y_1(x_0) - F(x_0)}{y_1(x_0)}\right| - \ln\left|\frac{y_2(x_0) - F(x_0)}{y_2(x_0)}\right| + \int_0^{x_0} \tilde{F}(x)dx$$

$$= \ln\left|1 + \frac{(y_1(x_0) - y_2(x_0))F(x_0)}{y_1(x_0)(y_2(x_0) - F(x_0))}\right| + \int_0^{x_0} \tilde{F}(x)dx,$$

where
$$\tilde{F}(x) = \frac{g(x)F(x)}{y_1(x)(F(x) - y_1(x))^2} - \frac{g(x)F(x)}{y_2(x)(F(x) - y_2(x))^2}$$

and $y_i(x)$ is the function corresponding to the arc $\widehat{A_i B_i}$. Clearly, since $F(x) < 0 < y_1(x) < y_2(x)$ in $(0, x_0)$ we obtain

$$\frac{(y_1(x_0) - y_2(x_0))F(x_0)}{y_1(x_0)(y_2(x_0) - F(x_0))} > 0, \quad \int_0^{x_0} \tilde{F}(x)dx > 0.$$

Thus

(21) $$\int_{\widehat{A_2 B_2}} f(x)dt > \int_{\widehat{A_1 B_1}} f(x)dt.$$

Similarly we obtain that

$$\int_{\widehat{C_2 F_2}} f(x)dt - \int_{\widehat{C_1 F_1}} f(x)dt > 0.$$

Setting $z = \int_0^x g(s)ds$, from (20) we get

$$\int_{\widehat{B_1 G_1 C_1}} F'(x)dt = \int_{\widehat{B_1 G_1 C_1}} \hat{F}'(z)dy = \int_{x_0}^{z_{G_1}} V(\hat{F}(z), \varphi_{G_1}(z), \tilde{\varphi}_{G_1}(z))dz.$$

By Lemma 7, in order to prove the inequality

$$\int_{z(x_0)}^{z_{G_2}} V(\hat{F}(z), \varphi_{G_2}(z), \tilde{\varphi}_{G_2}(z))dz > \int_{z(x_0)}^{z_{G_1}} V(\hat{F}(z), \varphi_{G_1}(z), \tilde{\varphi}_{G_1}(z))dz,$$

where $\hat{F}(z) = F(x) - F(x_0)$, we only need to prove that $\hat{F}(z(x_0)) = 0$, $\hat{F}(z) > 0$ and $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > z(x_0)$. Clearly we have $\hat{F}(z(x_0)) = 0$ and $\hat{F}(z) > 0$ for $z > z(x_0)$. Note that

$$\hat{F}(z)\hat{F}'(z) = [F(x) - F(\sqrt{1 - a^2})]f(x)/g(x).$$

For $x > \sqrt{1 - a^2}$ we have

$$\begin{aligned}
&\frac{[F(x) - F(\sqrt{1 - a^2})]f(x)}{g(x)} \\
&= \frac{\xi^2[bx + x^3 - b\sqrt{1 + a^2} - (1 - a^2)\sqrt{1 - a^2}](b + 3x^2)}{x(1 - 1/\sqrt{x^2 + a^2})} \\
&= \frac{\xi^2[b + x^2 + (1 - a^2) + \sqrt{1 - a^2}x](b + 3x^2)(\sqrt{x^2 + a^2} + x^2 + a^2)}{x(x + \sqrt{1 - a^2})} \\
&= \xi^2 \left(\frac{b + 1 - a^2}{x + \sqrt{1 - a^2}} + x \right) \left(\frac{b}{x} + 3x \right) (\sqrt{x^2 + a^2} + x^2 + a^2),
\end{aligned}$$

where the three factors of the last line are positive and increasing. Therefore, $[F(x) - F(\sqrt{1 - a^2})]f(x)/g(x)$ is positive and increasing. By Lemma 7, we have that

$$\int_{\widehat{B_2 G_2 C_2}} f(x)dt - \int_{\widehat{B_1 G_1 C_1}} f(x)dt > 0.$$

Therefore

(22) $$\oint_{L_1} \operatorname{div}(y - F(x), -g(x))dt > \oint_{L_2} \operatorname{div}(y - F(x), -g(x))dt.$$

Now we consider $b < 3a^2 - 3$. The corresponding phase portrait is shown in Figure 5(b). By the Bendixson criterium again, each L_i has two intersection points

with the straight line $x = x_0$, being x_0 the unique positive zero of $F'(x)$ when $x > 0$, and the intersection points are denoted by B_i and C_i $(i = 1, 2)$. Then the inequality (22) can be proved in a similar way to the case $3a^2 - 3 \le b < a^2 - 1$. Since $f(x) < 0$ for $0 < x < x_0$ and $y_i(x) - F(x) > 0$ $(i = 1, 2)$, we have

$$\int_{\widehat{A_2 B_2}} f(x)dt - \int_{\widehat{A_1 B_1}} f(x)dt = \int_0^{x_0} \left\{ \frac{f(x)}{y_2(x) - F(x)} - \frac{f(x)}{y_1(x) - F(x)} \right\} dx$$

$$(23) \qquad\qquad = \int_0^{x_0} \frac{f(x)(y_1(x) - y_2(x))}{(y_1(x) - F(x))(y_2(x) - F(x))} dx > 0.$$

Similarly we obtain that

$$\int_{\widehat{C_2 F_2}} f(x)dt - \int_{\widehat{C_1 F_1}} f(x)dt > 0.$$

Now, using again Lemma 7, we only need to prove that $\hat{F}(z(x_0)) = 0$, and $\hat{F}(z) > 0$, $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > z(x_0)$, where $\hat{F}(z) = F(x) - F(x_0)$. Clearly, we have $\hat{F}(z(x_0)) = 0$ and $\hat{F}(z) > 0$ for $z > z(x_0)$. Note that $\hat{F}(z)\hat{F}'(z) = [F(x) - F(\sqrt{-b/3})]f(x)/g(x)$. For $x > \sqrt{-b/3}$ we have

$$\frac{[F(x) - F(\sqrt{-b/3})]f(x)}{g(x)}$$
$$= \frac{\xi^2[bx + x^3 - b\sqrt{-b/3} - (-b/3)\sqrt{-b/3}](b + 3x^2)}{x(1 - 1/\sqrt{x^2 + a^2})}$$
$$= \frac{\xi^2[2b/3 + x^2 + \sqrt{-b/3}x](b + 3x^2)(\sqrt{x^2 + a^2} + x^2 + a^2)}{x(x + \sqrt{1 - a^2})} \frac{x - \sqrt{-b/3}}{x - \sqrt{1 - a^2}}$$
$$= \xi^2 \left(\frac{2b/3 - \sqrt{1 - a^2}(\sqrt{-b/3} - \sqrt{1 - a^2})}{x + \sqrt{1 - a^2}} + x + \sqrt{-b/3} - \sqrt{1 - a^2} \right)$$
$$\left(\frac{b}{x} + 3x \right)(\sqrt{x^2 + a^2} + x^2 + a^2)\left(1 + \frac{\sqrt{1 - a^2} - \sqrt{-b/3}}{x - \sqrt{1 - a^2}} \right),$$

where all of the factors of the last two lines are positive and increasing. Therefore $[F(x) - F(\sqrt{-b/3})]f(x)/g(x)$ is positive and increasing. By Lemma 7, we have

$$\int_{\widehat{B_2 G_2 C_2}} f(x)dt - \int_{\widehat{B_1 G_1 C_1}} f(x)dt > 0$$

and therefore (22) follows.

However, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium (equilibria) adjacent one to the other. So, from the inequality (22) and the repelling of the infinity, we obtain that system (2) has at most three large limit cycles, where the outer one is stable, the middle one is semistable, the inner one is stable. Clearly, for fixed values a and ξ, system (2) is a family of generalized rotated vector fields with respect to the parameter b. Assume that system (2) has exactly three large limit cycles. By Theorem 3.5 of [18, Chapter 4], the outer limit cycle and the inner one neither split, nor disappear as b varies monotonically. By Theorem 3.4 of [18, Chapter 4], the middle limit cycle will bifurcate into at least one stable and one unstable cycle when b varies in the suitable direction. This is a contradiction. Therefore, system (2) has at most two large limit cycles. If the two large limit cycles exist, we can obtain that the outer limit cycle is stable and the inner one is unstable. $\qquad\square$

We consider a generic Liénard system

$$
\begin{aligned}
\dot{x} &= y - \bar{F}(x), \\
\dot{y} &= -\bar{g}(x),
\end{aligned}
$$
(24)

where $\bar{F}$ is $\mathcal{C}^2$, $\bar{g}$ is $\mathcal{C}^1$ and $x \in (\alpha, \beta)(\alpha, \beta$ can be $\pm\infty)$. The following proposition partly improves the result of [9, Theorem 2.1].

Proposition 9. *Consider system* (24), *which satisfies* **(i)**-**(iv)**, *where*

(i): $\bar{f}(x) = \bar{F}'(x)$ *has a unique zero* 0 *and* $\bar{f}(x) < 0$ *(resp.* > 0*) as* $\alpha < x < 0$ *(resp.* $0 < x < \beta$*);*

(ii): $\bar{F}(0) = 0$*;*

(iii): $x\bar{g}(x) > 0$ *for* $x \neq 0$, $x \in (\alpha, \beta)$*;*

(iv): *the system*

$$
\bar{F}(x_1) = \bar{F}(x_2), \quad \bar{\lambda}(x_1) = \bar{\lambda}(x_2)
$$
(25)

has at most one solution $x_2 < 0 < x_1$, *where* $\bar{\lambda} := \bar{g}(x)/\bar{f}(x)$.

Moreover, when $\bar{F}(x_1) = \bar{F}(x_2)$, $\bar{\lambda}(x_1) > \bar{\lambda}(x_2)$ *(resp.* $\bar{F}(x_1) = \bar{F}(x_2)$, $\bar{\lambda}(x_1) < \bar{\lambda}(x_2)$*) as* $|x_1|, |x_2|$ *are small, system* (24) *satisfies either* **(v)** *or* **(v′)**, *where*

(v): *the function* $\bar{F}(x)\bar{f}(x)/\bar{g}(x)$ *is decreasing (resp. increasing) for* $\alpha < x < 0$*;*

(v′): *the function* $\bar{F}(x)\bar{f}(x)/\bar{g}(x)$ *is increasing (resp. decreasing) for* $0 < x < \beta$ *and*

$$
\lim_{x \to \alpha^+} \bar{F}(x) = \lim_{x \to \beta^-} \bar{F}(x).
$$

Then system (24) *has at most one closed orbit in the region* $\{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\}$. *The closed orbit is simple and unstable (resp. stable) if it exists.*

Proof. Assume that system (24) exists a limit cycle γ, as shown in Figure 6(a). In the following we will ascertain the sign of

$$
-\oint_\gamma \bar{f}(x)dt = \oint_\gamma \mathrm{dvi}(y - \bar{F}(x), -\bar{g}(x))dt.
$$

Clearly $w = \bar{F}(x)$ has two inverse functions, $x_1(w)$ (respectively $x_2(w)$), on the right (resp. left) side of the origin. The functions $\bar{\lambda}(x_i(w))$ will be denoted simply by $\lambda_i(w)$.

By $w = \bar{F}(x)$, we rewrite system (24) into

$$
\dot{w} = \bar{f}(x_i(w))(y - w), \quad \dot{y} = -\bar{g}(x_i(w)),
$$
(26)

which deduces

$$
\frac{dy}{dw} = \frac{\lambda_i(w)}{w - y}.
$$
(27)

Let $y_1(w)$ and $y_2(w)$(resp. $z_1(w)$ and $z_2(w)$) be functions determined by the orbits of (26) below (resp. above) the line $y = w$, which correspond the parts of the trajectories of system (24) below (resp. above) the curve $y = \bar{F}(x)$ and depend whether they are to the left or right of the origin.

From condition **(iv)**, $\lambda_1(w) = \lambda_2(w)$ has at most one root. When the equation $\lambda_1(w) = \lambda_2(w)$ has no roots, by the comparison theorem we obtain either $z_1(w) >$
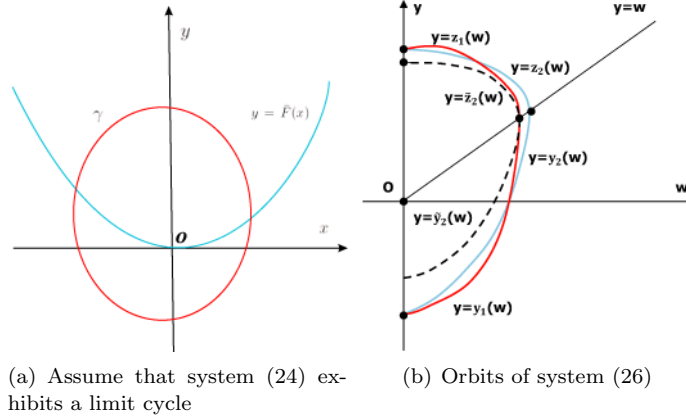
(a) Assume that system (24) ex-   (b) Orbits of system (26)
hibits a limit cycle

FIGURE 6. Limit cycles of system (24).

$z_2(w)$ and $y_1(w) < y_2(w)$ or $z_1(w) < z_2(w)$ and $y_1(w) > y_2(w)$. By Green formula, we have

$$\oint_\gamma \bar{g}(x)dx + (y - \bar{F}(x))dy = -\iint_{\text{Int}\gamma} \bar{f}(x)dxdy = \iint_{\text{Int}\gamma,x<0} dwdy - \iint_{\text{Int}\gamma,x>0} dwdy \neq 0,$$

which contradicts $\oint_\gamma \bar{g}(x)dx + (y - \bar{F}(x))dy = 0$. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root and the equation $z_1(w) = z_2(w)$ has no roots, we can similarly prove that (24) has no limit cycles. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root, applying again the comparison theorem the equation $z_1(w) = z_2(w)$ has at most one root.

Now we only need to discuss the system $\lambda_1(w) = \lambda_2(w)$ and $z_1(w) = z_2(w)$ having a unique root. Here we consider that condition **(v)** holds. By $(w, y) \to (\mu w, \mu y)$ with $\mu := w_2^*/w_1^* > 1$, where w_i^* is the intersection of $z_i(w)$ and the line $y = w$, system (27) deduces

$$(28) \qquad \frac{dy}{dw} = \frac{\lambda_2(\mu w)}{\mu(w - y)}.$$

First, we consider that $\lambda_1(w) > \lambda_2(w)$ as $w \to 0$. Moreover, the limit cycle γ of (24) with $x > 0$ corresponds to the broken curve in Figure 6(b) and $y_2(w)$, $z_2(w)$ are changed into two new functions, denoted $\tilde{y}_2(w)$ and $\tilde{z}_2(w)$, respectively. It is easy to check that $y_2(w) = \mu\tilde{y}_2(w/\mu)$ and $z_2(w) = \mu\tilde{z}_2(w/\mu)$. By $\mu > 1$ and the increasing of $\bar{\lambda}(w)/w$, we get $\bar{\lambda}(\mu w)/\mu > \bar{\lambda}(w)$. Furthermore, using again the comparison theorem to (27) and (28), we obtain $y_1(w) < \tilde{y}_2(w)$ and $z_1(w) > \tilde{z}_2(w)$.

Thus

$$\oint_\gamma \bar{f}(x)dt = \oint_\gamma \frac{\bar{f}(x)}{y - \bar{F}(x)}dx$$

$$= \int_0^{w_2^*} \frac{dw}{y_2(w) - w} - \int_0^{w_2^*} \frac{dw}{z_2(w) - w} + \int_0^{w_1^*} \frac{dw}{z_1(w) - w} - \int_0^{w_1^*} \frac{dw}{y_1(w) - w}$$

$$= \int_0^{w_1^*} \frac{\mu dw}{y_2(\mu w) - \mu w} - \int_0^{w_1^*} \frac{\mu dw}{z_2(\mu w) - \mu w} + \int_0^{w_1^*} \frac{dw}{z_1(w) - w} - \int_0^{w_1^*} \frac{dw}{y_1(w) - w}$$

$$= \int_0^{w_1^*} \frac{(y_1 - \tilde{y}_2)dw}{(y_1(w) - w)(\tilde{y}_2(w) - w)} - \int_0^{w_1^*} \frac{(z_1 - \tilde{z}_2)dw}{(z_1(w) - w)(\tilde{z}_2(w) - w)} < 0.$$

So in the region $\{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\}$ γ is unstable and simple if it exists. Moreover, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium adjacent one to the other. Therefore, the uniqueness has also been proved.

For the case $\lambda_1(w) < \lambda_2(w)$ as w is small, we can prove that $\oint_\gamma f(x)dt > 0$ in a similar way to the case $\lambda_1(w) > \lambda_2(w)$. So, in the region $\{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\}$, the limit cycle γ is stable and simple if it exists. Therefore, we have completed this proof. □

The proof of Proposition 9 gives the following corollary directly.

Corollary 10. *Assume that system* (24) *satisfies conditions* **(i)**-**(iii)** *of Proposition 9. If there is no solutions to* (25), *then system* (24) *has no closed orbits.*

Under the preparations of above Proposition 9 and Corollary 10, we obtain the existence of small limit cycles on the parameter surfaces H_2 and H_3 as follows.

Lemma 11. *Assume that* $0 < a < 1$ *and* $b = 3a^2 - 3$. *Then system* (2) *has no small limit cycles if* $1/\sqrt{3} \leq a < 1$, *and at most two small limit cycles if* $0 < a < 1/\sqrt{3}$.

Proof. By the transformation $(x, y) \to (x + \sqrt{1 - a^2}, y + F(\sqrt{1 - a^2}))$, system (2) can be rewritten as

$$(29) \qquad \begin{aligned} \dot{x} &= y - F(x + \sqrt{1 - a^2}) + F(\sqrt{1 - a^2}), \\ \dot{y} &= -g(x + \sqrt{1 - a^2}). \end{aligned}$$

It is easy to show that system (29) satisfies conditions **(i)**-**(iii)** of Proposition 9 when $x \in (-\sqrt{1 - a^2}, +\infty)$. Condition **(iv)** of Proposition 9 is equivalent to the fact that system

$$(30) \qquad \begin{aligned} F(\tilde{x}_1 + \sqrt{1 - a^2}) &= F(\tilde{x}_2 + \sqrt{1 - a^2}), \\ \frac{g(\tilde{x}_1 + \sqrt{1 - a^2})}{f(\tilde{x}_1 + \sqrt{1 - a^2})} &= \frac{g(\tilde{x}_2 + \sqrt{1 - a^2})}{f(\tilde{x}_2 + \sqrt{1 - a^2})}, \end{aligned}$$

has at most one solution, where $-\sqrt{1 - a^2} < \tilde{x}_1 < 0 < \tilde{x}_2$. Clearly (30) is equivalent to the fact that the system

$$(31) \qquad F(x_1) = F(x_2), \qquad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}$$

has a unique solution when $0 < x_1 < \sqrt{1 - a^2} < x_2$ and $x_j = \tilde{x}_j + \sqrt{1 - a^2}$, $j = 1, 2$. Let $s := x_1 + x_2$. From $F(x_1) = F(x_2)$, we have $x_1 x_2 = 3(a^2 - 1) + s^2$.

Since $0 < x_1 < \sqrt{1-a^2} < x_2$, we obtain $\sqrt{3-3a^2} < s < 2\sqrt{1-a^2}$. Note that

(32)
$$\frac{g(x)}{f(x)} = \frac{x}{3\xi(\sqrt{x^2+a^2}+x^2+a^2)}.$$

From the second equality of (31) and (32), we have

$$s^2 + 2a^2 - 3 - \frac{a^2 s}{x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}} = 0.$$

And consequently

$$\begin{aligned}
\left(x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}\right)^2 &= 2x_1^2x_2^2 + a^2(x_1^2+x_2^2) + 2x_1x_2\sqrt{x_1^2x_2^2+a^2(x_1^2+x_2^2)}+a^4 \\
&= 2s^4 + (11a^2-12)s^2 + 6(1-a^2)(3-2a^2) \\
&\quad +2(s^2+3a^2-3)\sqrt{s^4+(5a^2-6)s^2+(2a^2-3)^2}.
\end{aligned}$$

Let $\eta := s^2$. Then, for $3(1-a^2) < \eta < 4(1-a^2)$, we have

$$\begin{aligned}
&\frac{d\left[\left(x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}\right)^2/s^2\right]}{d\eta} \\
=\ & 2 - \frac{6(1-a^2)(3-2a^2)}{\eta^2} + \frac{6(1-a^2)}{\eta^2}\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2} + \frac{(\eta+3a^2-3)(2\eta+5a^2-6)}{\eta\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}} \\
=\ & 2 + \frac{6(1-a^2)(\eta+5a^2-6)}{\eta(\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}+3-2a^2)} + \frac{(\eta+3a^2-3)(2\eta+5a^2-6)}{\eta\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}} \\
>\ & 2 - \frac{6(1-a^2)}{\eta} - \frac{\eta+3a^2-3}{\eta} > 0.
\end{aligned}$$

Thus, if $h(\eta) := \eta + 2a^2 - 3 - a^2\sqrt{\eta}/[x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}]$, then the function $h(\eta)$ is increasing in $3(1-a^2) < \eta < 4(1-a^2)$. On the other hand, we have $h(3-3a^2) = -a-a^2 < 0$ and $h(4-4a^2) = 1-3a^2$. Therefore, when $\sqrt{3}/3 \leq a < 1$, the function $h(\eta)$ has no solutions for $3(1-a^2) < \eta < 4(1-a^2)$. By Corollary 10, system (2) has no limit cycles in the region $x > 0$. When $0 < a < \sqrt{3}/3$, the function $h(\eta)$ has a unique root in $3(1-a^2) < \eta < 4(1-a^2)$. Now we only need to verify the condition $(\mathbf{v'})$ of Proposition 9. We have

$$\begin{aligned}
\frac{(F(x)-F(\sqrt{1-a^2}))f(x)}{g(x)} &= 3\xi^2\frac{\sqrt{x^2+a^2}+x^2+a^2}{x}\left((3a^2-3)x+2(1-a^2)^{3/2}+x^3\right) \\
&= 3\xi^2\left(3a^2-3+\frac{2(1-a^2)^{3/2}}{x}+x^2\right)(\sqrt{x^2+a^2}+x^2+a^2).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\frac{d[(F(x)-F(\sqrt{1-a^2}))f(x)/(3\xi^2g(x))]}{dx} &= \left(\frac{-2(1-a^2)^{3/2}}{x^2}+2x\right)(\sqrt{x^2+a^2}+x^2+a^2) \\
&\quad + \left(3a^2-3+\frac{2(1-a^2)^{3/2}}{x}+x^2\right)\left(\frac{x}{\sqrt{x^2+a^2}}+2x\right) > 0,
\end{aligned}$$

for $0 < a < \sqrt{3}/3$ and $x > \sqrt{1 - a^2}$, because

$$\frac{-2(1 - a^2)^{3/2}}{x^2} + 2x > 0 \text{ and}$$

$$3a^2 - 3 + \frac{2(1 - a^2)^{3/2}}{x} + x^2 = \frac{(x - \sqrt{1 - a^2})(x^2 + \sqrt{1 - a^2}x - 2(1 - a^2))}{x} > 0.$$

So condition $(\mathbf{v'})$ of Proposition 9 holds. Therefore, when $0 < a < \sqrt{3}/3$, system (2) has at most one limit cycle which lies in the region $x > 0$ by Proposition 9. $\quad\square$
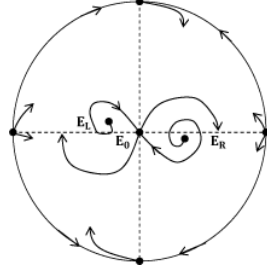

FIGURE 7. The uniqueness of limit cycles.

Lemma 12. *When $1/\sqrt{3} \le a < 1$ and $b = 3a^2 - 3$, system (2) has a unique limit cycle surrounding all equilibria.*

Proof. If $1/\sqrt{3} \le a < 1$ and $b = 3a^2 - 3$, i.e., on the parameter curve H_2 we obtain Figure 7 by Proposition 4 and Lemma 11, which shows the existence of a Poincaré-Bendixson annulus, i.e., any trajectory starting at a point of the boundary curves of the annulus enters (or leaves) the annulus, and inside the annulus there is no equilibrium points. So, the existence of some large limit cycles is obtained. Assume that system (2) has two large limit cycles. Let the outer limit cycle and inner one denoted by γ_2 and γ_1. Therefore $\oint_{\gamma_i} \text{div}(y - F(x), -g(x))dt \le 0$ for $i = 1, 2$. By (22), $\oint_{\gamma_1} \text{div}(y - F(x), -g(x))dt > \oint_{\gamma_2} \text{div}(y - F(x), -g(x))dt$. However, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium (equilibria) adjacent one to the other. Therefore

$$\oint_{\gamma_1} \text{div}(y - F(x), -g(x))dt = 0 > \oint_{\gamma_2} \text{div}(y - F(x), -g(x))dt.$$

So γ_1 is a semistable limit cycle. Since the vector field of (2) is rotating with respect to b, by Theorem 3.4 of [18, p.211] there is a stable limit cycle $\hat{\gamma}_2$ near γ_2 for a perturbation of b, and two limit cycles $\hat{\gamma}_1, \tilde{\gamma}_1$ ($\hat{\gamma}_1$ is smaller than $\tilde{\gamma}_1$) near γ_1. By (22) and stabilities of equilibria, we obtain

$$\oint_{\hat{\gamma}_1} \text{div}(y - F(x), -g(x))dt \le 0, \qquad \oint_{\tilde{\gamma}_1} \text{div}(y - F(x), -g(x))dt \ge 0,$$

and

$$\oint_{\hat{\gamma}_2} \text{div}(y - F(x), -g(x))dt < 0,$$

which contradicts $\oint_{\hat{\gamma}_1} \text{div}(y - F(x), -g(x))dt > \oint_{\tilde{\gamma}_1} \text{div}(y - F(x), -g(x))dt$. There-fore, system (2) has at most one large limit cycle. Thus, the uniqueness of limit cycle is proved and the phase portrait (d) of Figure 2 in Theorem 1 is obtained. $\square$

Moreover, we can get the phase portraits (i), (e) and (n) of Figure 2 in Theorem 1 by Proposition 3, Lemmas 8, 11, 12 and the continuity of the vector fields.

From [5] it follows that system (2) has no limit cycles for $a \to 0$ and $b = 3a^2 - 3$.

Proposition 13. *System* (2) *has at most two small limit cycles around the equi-libria* E_R *or* E_L *for the values of the parameters in* $\mathcal{G}_1 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < 1, \ b < 3a^2 - 3, \ 0 < \xi \ll 1\}$.

Proof. By the homeomorphism $(x, y) \to (x, y + F(x))$, system (2) can be rewritten as

$$(33) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) - \xi(b + 3x^2)y. \end{aligned}$$

In $x > 0$ we do the change of variables $w = \sqrt{x^2 + a^2}$ and the time scaling $dt = \sqrt{1 + a^2/x^2}d\tau$ to system (33), and we get

$$(34) \qquad \begin{aligned} \dot{w} &= y, \\ \dot{y} &= 1 - w - \frac{\xi w}{\sqrt{w^2 - a^2}}(b - 3a^2 + 3w^2)y. \end{aligned}$$

When $\xi = 0$ this system is the Hamiltonian system

$$(35) \qquad \begin{aligned} \dot{w} &= y, \\ \dot{y} &= 1 - w, \end{aligned}$$

with the first integral

$$(36) \qquad H(w, y) = \frac{y^2}{2} + \frac{w^2}{2} - w.$$

Its level curves $\Gamma_h := \{(w, y) : \ H(w, y) = h, \ -1/2 \leq h < a^2/2 - a\}$ are shown in Figure 8.
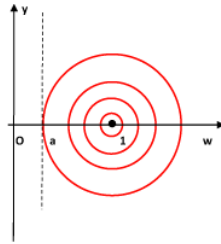

FIGURE 8. The phase portrait of the Hamiltonian system (35).

Of course $H = -1/2$ corresponds to the center $(1, 0)$, and for the values of h such that $-1/2 < h < a^2/2 - a$, the curve $H(w, y) = h$ corresponds to a periodic orbit of Hamiltonian system (35) surrounding the point $(1, 0)$, which intersects the positive half w-axis inside the interval $(a, 1)$.

Now we consider system (34) as a perturbation of system (35) for small ξ. Here we only will discuss how many small limit cycles surround the equilibrium point

$(1,0)$ of system (34) when ξ is sufficiently small. On the other hand, for every $h \in (-1/2, a^2/2 - a)$ the orbit Γ_h intersects the the segment $L_1 : (a,1)$ of the w-axis at exactly one point $Q_h(x(h),0)$. Therefore, the segment L_1 can be parameterized by $h \in (-1/2, a^2/2 - a)$.

For every $h \in (-1/2, a^2/2 - a)$, we consider the trajectory of system (33) passing through the point $Q_h(x(h),0) \in L_1$. This trajectory goes forward and backward until it intersects the positive w-axis at points Q_1 and Q_2, respectively, as in Figure 9. We denote the piece of trajectory from Q_2 to Q_1 by $\gamma(h,\xi,a,b)$. Then $\gamma(h,\xi,a,b)$ is a periodic orbit if and only if $Q_1 = Q_2$. From (36), we have

$$\frac{\partial H(w,y)}{\partial w} = w - 1 \neq 0, \text{ if } |w| \neq 1.$$
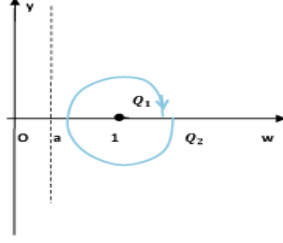
Hence $Q_1 = Q_2$ if and only if $H(Q_1) = H(Q_2)$.

FIGURE 9. The perturbation of (35).

On the other hand, along the orbits of system (35) we have

$$\frac{dH(w,y)}{dt} dt = -\frac{\xi w}{\sqrt{w^2 - a^2}}(b - 3a^2 + 3w^2)y^2 dt = -\frac{\xi w}{\sqrt{w^2 - a^2}}(b - 3a^2 + 3w^2)y\,dw.$$

This implies that

$$H(Q_1) - H(Q_2) = \int_{t(Q_2)}^{t(Q_1)} \left.\frac{dH(w,y)}{dt}\right|_{(35)} dt$$

$$= -\xi \int_{\gamma(h,\xi,a,b)} \frac{w}{\sqrt{w^2 - a^2}}(b - 3a^2 + 3w^2)y\,dw = -\xi F(h,\xi,a,b).$$

Therefore $\gamma(h,\xi,a,b)$ is a periodic orbit if and only if $F(h,\xi,a,b) = 0$.

We consider $F(h,\xi,a,b)$ as a perturbation of $F(h,0,a,b)$. The function $F(h,0,a,b)$ is given by

$$F(h,0,a,b) = (b - 3a^2)I_1(h) + 3I_3(h),$$

where

$$I_i(h) = \int_{\Gamma_h} \frac{w^i}{\sqrt{w^2 - a^2}} y\,dw \quad \text{for} \quad i = 1, 3.$$

The orientation of Γ_h is determined by the direction of the vector field (35). By the Green's formula

$$I_1(h) = \int_{\Gamma_h} \frac{w}{\sqrt{w^2 - a^2}} y\,dw = \iint_{D(h)} \frac{w}{\sqrt{w^2 - a^2}} dw\,dy > 0, \quad h \in (-1/2, a^2/2 - a),$$

where $D(h)$ is the region surrounded by Γ_h.

It is easy to show that

$$\lim_{h \to -1/2} I_1(h) = \lim_{h \to -1/2} I_3(h) = 0.$$

By the Mean Value Theorem of integrals, we have

$$\lim_{h \to -1/2} \frac{I_3(h)}{I_1(h)} = \lim_{h \to -1/2} \frac{\iint_{D(h)} \dfrac{w^3}{\sqrt{w^2 - a^2}} \, dw \, dy}{\iint_{D(h)} \dfrac{w}{\sqrt{w^2 - a^2}} \, dw \, dy} = \lim_{h \to -1/2} \bar{w}^2(h) = 1,$$

where $(\bar{w}(h), \bar{y}(h)) \in D(h)$ and $D(h)$ shrinks to the point $(1,0)$ as $h \to -1/2$.

Now we define the function

$$u(h) = \begin{cases} \dfrac{I_3(h)}{I_1(h)} & \text{if } h \in (-1/2, a^2/2 - a), \\[2mm] 1 & \text{if } h = -1/2. \end{cases}$$

It is continuous in the interval $[-1/2, a^2/2 - a]$. Therefore, to determine the existence and the number of small limit cycles for system (34) for ξ sufficiently small, we only need to study the existence and the number of zeros of the function $F(h, \xi, a, b)$ in the interval $h(-1/2, a^2/2 - a)$. In addition, since

$$F(h, \xi, a, b) \approx F(h, 0, a, b) = I_1(h) \left(b - 3a^2 + 3 \frac{I_3(h)}{I_1(h)} \right),$$

for ξ sufficiently small, we obtain that the behavior of the function $u = u(h)$, as a ratio of two Abelian integrals, is crucial in our discussion.

From the notations in [12, Corollary 2], taking

$$\Phi(w) = \frac{1}{2}w^2 - w, \ \Psi(y) = \frac{y^2}{2}, \ f_1(w) = \frac{w}{\sqrt{w^2 - a^2}}, \ f_3(w) = \frac{w^3}{\sqrt{w^2 - a^2}}$$

for system (34). Note that $H(w, y) = \Phi(w) + \Psi(y)$ and $I_i(h) = \int_{\Gamma_h} f_i(w) \, y \, dw$, $i = 1, 3$. We can calculate

$$M(\tilde{w}, w) + M(w, \tilde{w}) = 4(w - 1)^4 \left\{ \frac{-w^2(2 - w)^2 + a^2(4 - 2w + w^2)}{(w^2 - a^2)[(2 - w)^2 - a^2]} - \right.$$

$$\left. \frac{[w^3(2 - w)^3 + 2(-4 + 2w + 3w^2 - 4w^3 + w^4)a^2 + (2 + 2w - w^2)a^4]}{(w^2 - a^2)^{3/2}[(2 - w)^2 - a^2]^{3/2}} \right\},$$

where $\tilde{w} + w = 2$, and

$$M(\tilde{w}, w) = [\Phi'(\tilde{w})]^2 \{ \Phi'(\tilde{w})(f_3'(w)f_1(w) - f_1'(w)f_3(w)) + \Phi''(w)(f_3(w)f_1(\tilde{w})$$

$$- f_1(w)f_3(\tilde{w})) + \Phi'(w)[f_3(\tilde{w})f_1'(w) - f_1(\tilde{w})f_3'(w)] \}.$$

In fact, $s_1(w, a) := -w^2(2 - w)^2 + a^2(4 - 2w + w^2)$ is decreasing if $w \in (a, 1)$. Therefore, $\min\{s_1(w, a)\} = s_1(1, a) = -1 + 3a^2$, and $\max\{s_1(w, a)\} = s_1(a, a) = 2a^3 > 0$ as $w \in (a, 1)$. Therefore $s_1(w, a) > 0$ when $1/\sqrt{3} \le a < 1$, and $s_1(w, a) = 0$ has a unique root (denoted by w_1) when $0 < a < 1/\sqrt{3}$. On the other hand, $s_2(w, a) := -w^3(2 - w)^3 - 2(-4 + 2w + 3w^2 - 4w^3 + w^4)a^2 - (2 + 2w - w^2)a^4$ is also decreasing if $w \in (a, 1)$ because $\partial s_2(w, a)/\partial w < 0$. Hence $\min\{s_2(w, a)\} = s_2(1, a) = (3a^2 - 1)(1 - a^2)$, and $\max\{s_2(w, a)\} = s_2(a, a) = 4(a - 1)(a - 2)a^2 > 0$ as $w \in (a, 1)$. Similarly, $s_2(w, a) > 0$ when $1/\sqrt{3} \le a < 1$, and $s_2(w, a) = 0$ has a unique root (denoted by w_2) when $0 < a < 1/\sqrt{3}$. Therefore, $M(\tilde{w}, w) + M(w, \tilde{w}) >$

0 for all $w \in (a,1)$ when $1/\sqrt{3} \le a < 1$. Next, we have that $w_1 < w_2$, because $s_2(w_1, a) = 2(4 - 6w_1 + 3w_1^2)a^2 - 6a^4 > 2(4 - 6w_1 + 3w_1^2)a^2 - 2a^2 = 6(1 - w_1)^2 a^2 > 0$. So we only need to consider the zeros of $M(\tilde{w}, w) + M(w, \tilde{w})$ for $w \in (w_1, 1)$. Moreover,

$$\frac{d\{\sqrt{(w^2 - a^2)[(2 - w)^2 - a^2]}\, s_1(w, a)\}}{dw} = \left\{ \frac{(2w(2 - w) + a^2)(w^2 - a^2)((2 - w)^2 - a^2)}{\sqrt{(w^2 - a^2)[(2 - w)^2 - a^2]}} \right.$$
$$\left. - \frac{[w(2 - w) + a^2][-w^2(2 - w)^2 + a^2(4 - 2w + w^2)]}{\sqrt{(w^2 - a^2)[(2 - w)^2 - a^2]}} \right\} 2(w - 1) < 0,$$

for $w \in (w_1, 1)$. Therefore, $\sqrt{(w^2 - a^2)[(2 - w)^2 - a^2]}\, s_1(w, a) + s_2(w, a)$ is decreasing for $w \in (w_1, 1)$, which implies that $\sqrt{(w^2 - a^2)[(2 - w)^2 - a^2]}\, s_1(w, a) + s_2(w, a)$ has a unique zero. Thus $M(\tilde{w}, w) + M(w, \tilde{w}) = 0$ has a unique zero for $w \in (w_1, 1)$.

By [12, Corollary 2], the inequality $M(\tilde{w}, w) + M(w, \tilde{w}) > 0$ (resp. < 0) induces that $u'(h) < 0$ (resp. > 0). So $u'(h)$ has at most one zero. Then $F(h, 0, a, b)$ has at most two zeros because $b - 3a^2 < 0$. The proof is completed. □

4. Global bifurcation

The aim of this section is to show that the global bifurcation surfaces HL, DL_1, DL_2 of homoclinic loop and double limit cycles in Figure 1 exist and how they are located in the bifurcation diagram of system (2).

By the transformation $(x, y) \to (x, y + F(x))$, system (2) can be rewritten as system (33). Then For fixed values of a and ξ, system (2) is a rotational family of vector fields (see [18] for definitions and properties) with respect to the parameter b. This implies that when b increases unstable limit cycles increase and stable ones decrease in size. Furthermore, the double limit cycle that is stable in its outside part splits into a pair of limit cycles.

Lemma 14. *The surface DL_2 does not lie in $\mathcal{G}_2 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < 1,\ 3a^2 - 3 < b < a^2 - 1\}$ and system (2) has at most one small limit cycle around the equilibria E_R or E_L in $\mathcal{G}_2$.*

Proof. By Lemma 11 system (2) has no small limit cycles if $1/\sqrt{3} \le a < 1$ and $b = 3a^2 - 3$, and at most one small limit cycle around equilibria E_R or E_L if $0 < a < 1/\sqrt{3}$ and $b = 3a^2 - 3$. Therefore DL_2 cannot intersect with the curve $b = 3a^2 - 3$ except when $a = 1/\sqrt{3}$. Computing the trace at E_0 we obtain $\mathrm{tr}(J_0) = -b\xi > 0$, because $b < 0$ and $\xi > 0$. So the homoclinic loops have to be unstable if they exist by [7, Chapter 3, Theorem 3.3]. Assume that system (2) exhibits at least two small limit cycles surrounding E_R for $(a, b, \xi) \in \mathcal{G}_2$. For fixed a and ξ, we obtain that system (2) has a small semi-stable limit cycle Γ_0 when $b = b_0$ from the rotational properties of system (2) with respect to b. Now, given b, ξ and a perturbation $a \to a + \varepsilon$, there exists a solution $\varphi(t, x_0, y_0)$ for $t \in (0, T)$ which lies in the small neighborhood of Γ_0 from the continuous dependence of solutions on the parameters and initial conditions, where T is the periodic of Γ_0 and $\varepsilon > 0$ is small. Then using again the properties of the rotational vector fields, we can take a suitable parameter $b_0 + \varepsilon_1$ such that system (2) has a new semi-stable small limit cycle $\hat{\Gamma}_0$, because the homoclinic loops cannot be semistable, when $\varepsilon_1 > 0$ is small. By continuity system (2) has a small semi-stable limit cycle in the parameter curve either H_2, or

H_3, which leads to a contradiction. Therefore system (2) has at most one small limit cycle in $\mathcal{G}_2$. We have completed the proof. □

By a similar discussion as that in the proof of Lemma 14, the phase portraits (f)-(h) and (j)-(m) of system (2) in Figure 2 of Theorem 1 are obtained from the properties of rotational vector fields, continuity, Lemma 14 and the results of section 3.

Remark 1. *The surface DL_2 of double small limit cycles is the graph of a function* $b = \varphi_2(a, \xi)$.

From [5] we can obtain that a pair of grazing loop are stable for $a = 0$ if they exist. However, the pair of homoclinic loops have to be unstable if they exist when $a \neq 0$. In fact, HL, DL_1, DL_2 have a common intersection point for the limit value $a = 0$, i.e., $\varphi(0, \xi) = \phi_1(0, \xi) = \phi_1(0, \xi) < -3$. Since system (33) is a rotational vector field with respect to the parameter b, the manifolds of E_0 move monotonically as a, ξ are fixed and b increases, see [5, 6]. Therefore, it is worthwhile to note that HL, DL_1 and DL_2 have no intersection points except at endpoints. Summarizing the previous results, we can obtain Theorem 1, as shown in Figure 1.
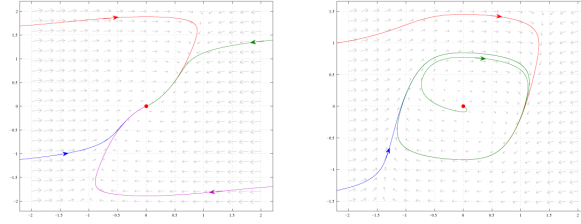
(a) $(\sqrt{2}, 1, 1) \in II$ (b) $(\sqrt{2}, -1, 1) \in III$

FIGURE 10. Simulations with a single equilibrium

5. NUMERICAL EXAMPLES

In this section we give several numerical examples of previous results.

Example 1. Let $a = \sqrt{2}$ and $\xi = 1$. When $b = 1$ the system has a unique equilibrium $(0, 0)$, which is a sink, and no limit cycles, as shown in Figure 10(a).

However, when $b = -1$ the system has a unique equilibrium, the origin $(0, 0)$ which is a source. Furthermore, from Lemma 5, there is a unique limit cycle, which is stable, as shown in Figure 10(b).

Example 2. Let $\xi = 1$. When $a = \sqrt{2}/2$ and $b = -2$ the system has three equilibria and exactly one large limit cycle, as shown in Figure 11(a).

When $a = 0.3$ and $b = -2.77$ the system has three equilibria and exactly two small limit cycles, as shown in Figure 11(b).

When $a = \sqrt{2}/2$ and $b = -0.5$ the system has three equilibria and no limit cycles, as shown in Figure 11(c).
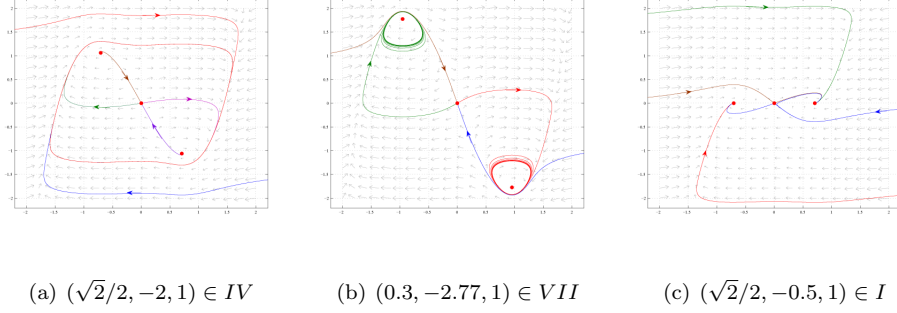
(a) $(\sqrt{2}/2, -2, 1) \in IV$ (b) $(0.3, -2.77, 1) \in VII$ (c) $(\sqrt{2}/2, -0.5, 1) \in I$

FIGURE 11. Simulations with three equilibria

(a) $(0.9, -0.46, 0.1) \in VIII$ (b) $(0.9, -0.5558, 0.1) \in IX$

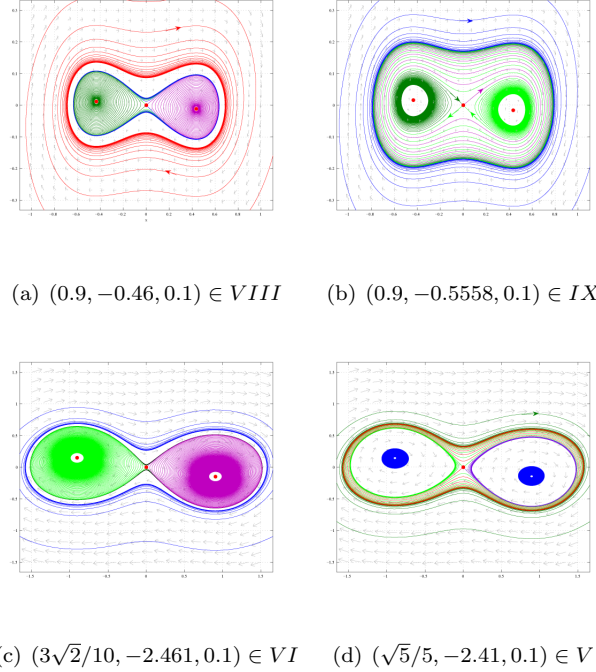(c) $(3\sqrt{2}/10, -2.461, 0.1) \in VI$ (d) $(\sqrt{5}/5, -2.41, 0.1) \in V$

FIGURE 12. Simulations with three equilibria

Example 3. Let $\xi = 0.1$. When $a = 0.9$ and $b = -0.46$ the system has three equilibria and exactly two large limit cycles, as shown in Figure 12(a).

When $a = 0.9$ and $b = -0.555$ the system has three equilibria, exactly two small limit cycles and one large limit cycle, as shown in Figure 12(b).

When $a = 3\sqrt{2}/10$ and $b = -2.461$ the system has three equilibria, exactly two small limit cycles and two large limit cycles, as shown in Figure 12(c).

When $a = \sqrt{5}/5$ and $b = -2.41$ the system has three equilibria, exactly four small limit cycles and one large limit cycle, as shown in Figure 12(d).

APPENDIX: AVERAGING THEORY OF FIRST AND SECOND ORDER

The averaging theory of second order for studying specifically periodic orbits can be found in [13] for $\mathbb{S}^3$ differential systems and in [1] for Lipschitz differential systems, see also Chapter 11 of [17]. Here we present a brief summary with the results that we need for studying the Hopf bifurcation of the SD oscillator systems.

Consider the differential system

(37) $$\dot{\mathbf{x}}(t) = \mu F_1(t, \mathbf{x}) + \mu^2 F_2(t, \mathbf{x}) + \mu^3 R(t, \mathbf{x}, \mu),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\mu_0, \mu_0) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume:

(i) $F_1(t, \cdot) \in \mathbb{S}^2(D)$, $F_2(t, \cdot) \in \mathbb{S}^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R, $D_{\mathbf{x}}^2 F_1$, $D_{\mathbf{x}} F_2$ are locally Lipschitz with respect to $\mathbf{x}$, and R is twice differentiable with respect to μ.

We define the functions $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as follows

$$f_1(\mathbf{x}) = \frac{1}{T} \int_0^T F_1(s, \mathbf{x}) ds,$$

$$f_2(\mathbf{x}) = \frac{1}{T} \int_0^T \left[D_{\mathbf{x}} F_1(s, \mathbf{x}) \int_0^s F_1(t, \mathbf{x}) dt + F_2(s, \mathbf{x}) \right] ds.$$

(ii) For $V \subset D$ an open and bounded set and for each $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$, suppose that either $f_1(\mathbf{x}) \not\equiv 0$, there exists $a \in V$ such that $f_1(a) = 0$ and the Jacobian $\det D_{\mathbf{x}}(f_1)(a) \neq 0$; or $f_1(\mathbf{x}) \equiv 0$, there exists $a \in V$ such that $f_2(a) = 0$ and the Jacobian $\det D_{\mathbf{x}}(f_2)(a) \neq 0$.

Then for $|\mu| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(t, \mu)$ of system (37) such that $\mathbf{x}(0, \mu) \mapsto a$ when $\mu \mapsto 0$.

If for the i for which $f_i(a) = 0$ the real part of all the eigenvalues of the Jacobian matrix $D_{\mathbf{x}}(f_i)(a)$ are negative, then the periodic solution $\mathbf{x}(t, \mu)$ is asymptotically stable, if some eigenvalue has a positive real part then it is unstable.

The *averaging theory of first order* takes place when $f_1(\mathbf{x}) \not\equiv 0$. If $f_1(\mathbf{x}) \equiv 0$ and $f_2(\mathbf{x}) \not\equiv 0$ we say that that we work with the *averaging theory of second order*.

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