

# Restricted independence in displacement function for better estimation of cyclicity <sup>\*</sup>

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## Abstract

Since the independence of focal values is a sufficient condition to give a number of limit cycles arising from a center-focus equilibrium, in this paper we consider a restricted independence to a parametric curve, which gives a method not only to increase the lower bound for the cyclicity of the center-focus equilibrium but also to be available when those focal values are not independent. We apply the method to a nondegenerate cubic center-focus variety and prove that the cyclicity reaches its an upper bound.

**Keywords:** center-focus variety; cyclicity; focal value; independence; power sequence.

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## 1 Introduction

As it appears in the books [3, 5, 19] and articles [4, 6, 13, 14, 18, 20], the discussion on the center-focus equilibria is one of the most important problems in ordinary differential equations. A center-focus equilibrium is an equilibrium at which the linear part of the differential system has a pair of nonvanished pure imaginary eigenvalues. The main interest on the research of these equilibria is the determination of the kind

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of their stability and on their *cyclicity* at this equilibrium, i.e., the greatest number of limit cycles which may arise from a Hopf bifurcation at these equilibria.

We generally consider the family of analytic systems

$$\dot{x} = \alpha x - y + P(x, y, \boldsymbol{\lambda}), \quad \dot{y} = x + \alpha y + Q(x, y, \boldsymbol{\lambda}) \quad (1.1)$$

with a standardized linear part, where  $\alpha \in \mathbb{R}$  and  $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$  are parameters,  $P(x, y, \boldsymbol{\lambda})$  and  $Q(x, y, \boldsymbol{\lambda})$  are real analytic functions of  $x$  and  $y$  starting at least with terms of degree two and depending polynomially on  $\boldsymbol{\lambda}$ . This family, denoted by  $LC(\alpha, \boldsymbol{\lambda})$  for short, is referred as a family of linear centers, which has either a center or a focus at the origin  $O : (0, 0)$  clearly. Its *center-focus variety* is

$$\mathcal{CF} := \{(\alpha, \boldsymbol{\lambda}) \in \mathbb{R} \times \mathbb{R}^m : \alpha = 0\} \cong \mathbb{R}^m.$$

In this variety the so-called center-focus equilibrium  $O$  needs to be identified between focus (called a *weak focus*) and center, which is decided if using finitely many focal values (see [3] for more details). Focal values come from the coefficients of the displacement function  $\Pi(\rho) := h(\rho) - \rho$ , where  $h$  is the Poincaré return map

$$h(\rho) := e^{2\pi\alpha}\rho + \sum_{i=2}^{+\infty} g_i(\alpha, \boldsymbol{\lambda})\rho^i \quad (1.2)$$

and  $g_i$ 's are analytic functions of  $\alpha$  and  $\boldsymbol{\lambda}$  such that  $g_2(0, \boldsymbol{\lambda}) \equiv 0$ . Let  $g_i(\boldsymbol{\lambda}) := g_i(0, \boldsymbol{\lambda})$  for all  $i = 2, 3, \dots$ . Since  $P, Q$  are assumed to be polynomially dependent on  $\boldsymbol{\lambda}$ ,  $g_i \in \mathbb{R}[\boldsymbol{\lambda}]$  (the ring of real polynomials in the variable  $\boldsymbol{\lambda}$ ) for all  $i \geq 2$  and  $g_{2k} \in \langle g_3, \dots, g_{2k-1} \rangle$  (the ideal generated by  $g_3, \dots, g_{2k-1}$  over the ring  $\mathbb{R}[\boldsymbol{\lambda}]$ ) for all  $k \geq 2$ , as indicated in [1, 3]. Those  $g_{2i+1}$ 's are called the *focal values*, which are algebraically equivalent to the Lyapunov quantities ([15, 16]). Note that  $\mathcal{CF}$  contains the subset

$$\mathcal{C} := V(g_3, \dots, g_{2i+1}, \dots),$$

where  $V(g_3, \dots, g_{2i+1}, \dots)$  denotes the algebraic variety of  $\langle g_3, \dots, g_{2i+1}, \dots \rangle$  and, by [7, p. 3], is actually the set of all common zeros of all  $g_{2i+1}$ 's ( $i \geq 1$ ). As in [3, p. 11],  $\mathcal{C}$  is called the *center variety* of the family  $LC(\alpha, \boldsymbol{\lambda})$  because the center-focus  $O$  of system  $LC(0, \boldsymbol{\lambda}')$  is a *center* if and only if  $\boldsymbol{\lambda}' \in \mathcal{C}$ . For any  $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$  there exists an integer  $k \geq 1$  such that  $g_{2k+1}(\boldsymbol{\lambda}') \neq 0$  and  $g_i(\boldsymbol{\lambda}') = 0$  for all  $i = 2, \dots, 2k$ , for which we call  $O$  a *weak focus of multiplicity  $k$*  in system  $LC(0, \boldsymbol{\lambda}')$ .

As usual the *cyclicity* of a center-focus is the maximal number of limit cycles emerging from it in the phase portrait when we change slightly the parameters of the system (see [3, 8, 10, 17] and references therein). More precisely, the greatest number of limit cycles bifurcated from  $O$  is called the *cyclicity* of system  $LC(0, \boldsymbol{\lambda}')$  at  $O$  (perturbed within the family  $LC(\alpha, \boldsymbol{\lambda})$ ) and denoted by  $\mathcal{N}(\boldsymbol{\lambda}')$ . In particular,  $\mathcal{N}(\boldsymbol{\lambda}')$  is denoted by  $\mathcal{N}_c(\boldsymbol{\lambda}')$  (resp.  $\mathcal{N}_f(\boldsymbol{\lambda}')$ ) and called *center cyclicity* (resp. *focus cyclicity*) of system  $LC(0, \boldsymbol{\lambda}')$  if  $\boldsymbol{\lambda}' \in \mathcal{C}$  (resp.  $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$ ).

The cyclicity  $\mathcal{N}(\lambda')$  is decided not only by the multiplicity of the center-focus equilibrium, but also by the greatest number of independent sign changes in the displacement function near  $(0, \lambda')$ . For multi-parametric families there are more difficulties in finding the greatest number of nonvanished focal values, but one usually gives its a lower bound. For such an independence, a well-known method is to check the following conditions:

(ID<sub>k</sub>-1) every neighborhood of  $\lambda'$  contains a  $\mu' \in V(g_3, \dots, g_{2k-1})$  such that  $g_{2k+1}(\mu') \neq 0$ , and

(ID<sub>k</sub>-2) for each positive integer  $\ell \leq k-1$  and each  $\mu' \in V(g_3, \dots, g_{2\ell+1})$  satisfying that  $g_{2\ell+3}(\mu') \neq 0$ , every neighborhood of  $\mu'$  contains a  $\mu'' \in V(g_3, \dots, g_{2\ell-1})$  such that  $g_{2\ell+1}(\mu'')g_{2\ell+3}(\mu') < 0$ .

Note that there is only (ID<sub>1</sub>-1) if  $k = 1$ . As indicated in [2], (ID<sub>k</sub>-1) and (ID<sub>k</sub>-2) are known as conditions for the first  $k$  focal values  $g_{2j+1}$ ,  $j = 1, \dots, k$ , to be independent, under which  $\mathcal{N}(\lambda') \geq k$  as shown in [12]. Another method ([3]) is to determine the rank  $r(\lambda')$  of the Jacobian matrix

$$\left. \frac{\partial(g_3, \dots, g_{2j+1})}{\partial \lambda} \right|_{\lambda=\lambda'},$$

where all  $g_3, \dots, g_{2j+1}$  vanish at  $\lambda'$ , which asserts  $\mathcal{N}(\lambda') \geq r(\lambda')$  because it implies the existence of an independent subsequence of  $r(\lambda')$  members in  $\{g_3, \dots, g_{2j+1}\}$  as indicated in ([2]). However, it is not easy to verify the independence of focal values or compute the rank of the Jacobian with many parameters. Besides, conditions (ID<sub>k</sub>-1) and (ID<sub>k</sub>-2) are strong sufficient conditions for the independence, which remind us to find weaker ones. The rank of the Jacobian gives a lower bound for  $\mathcal{N}(\lambda')$ , but this bound may not be the best.

In this paper we give a method to increase the lower bound for the cyclicity  $\mathcal{N}(\lambda')$  of system (1.1). The method is to find an appropriate curve passing through  $(0, \lambda')$  in the space  $\mathbb{R}^{m+1}$  of parameters  $(\alpha, \lambda)$ , on which those focal values depend on a single variable in such a way that we can determine easily the number of independent sign changes in the displacement function on the curve, called an *restricted independence* to the curve. Such a restriction may give a larger number of independent sign changes than the rank of the Jacobian, from which we can find more limit cycles bifurcating from the origin  $O$ . The result about this method is given in section 2. In section 3 we give some corollaries for easier applications, and practical application of our method with an example. This example has a nonvanished 4th order focal value but does not satisfy the independence condition of focal values, from which one cannot assert that the cyclicity of  $O$  is 4. However, using our method, we prove that the cyclicity is exactly 4. Finally, in section 4 we apply the method to a five-parametric family of cubic systems for finding its  $\mathcal{N}(\lambda')$ .

## 2 Restricted independence

For  $\lambda' \in \mathbb{R}^m \setminus \mathcal{C}$  the equilibrium  $O$  is a weak focus of system  $LC(0, \lambda')$ . Let  $\zeta(\lambda')$  be the multiplicity of the weak focus. Then  $\zeta(\lambda')$  gives an upper estimate for  $\mathcal{N}_f(\lambda')$  because

$$g_{2\zeta(\lambda') + 1}(\lambda') \neq 0, \quad \text{and} \quad g_i(\lambda') = 0 \quad \forall i < 2\zeta(\lambda') + 1, \quad (2.1)$$

and the sequence of  $g_{2k+1}$ 's ( $k = 1, \dots, \zeta(\lambda')$ ) may not be independent. In contrast, for  $\lambda' \in \mathcal{C}$ , equilibrium  $O$  is a center of system  $LC(0, \lambda')$ . Since  $\mathbb{R}\{\lambda\}_{\lambda'}$ , the ring of convergent power series at  $\lambda'$ , is a Noetherian ring (see [9, p. 147]), every ideal in this ring is finitely generated, which implies the existence of a least integer  $\iota(\lambda') > 0$  satisfying that

$$\langle g_3, g_5, \dots, g_{2\iota(\lambda') + 1} \rangle_{\lambda'} = \langle g_3, \dots, g_{2i+1}, \dots \rangle_{\lambda'} \quad \text{in } \mathbb{R}\{\lambda\}_{\lambda'}. \quad (2.2)$$

The integer  $\iota(\lambda')$ , called the *multiplicity of the center*  $O$ , gives an upper estimate for  $\mathcal{N}_c(\lambda')$ . In general, we see that the field  $\mathbb{R}$  is a commutative Noetherian ring, which implies by the Hilbert Basis Theorem ([9, p. 144]) that  $\mathbb{R}[\lambda]$  is a Noetherian ring. Therefore, there exists the least integer  $\iota_p \geq 1$  such that

$$\langle g_3, g_5, \dots, g_{2\iota_p + 1} \rangle = \langle g_3, \dots, g_{2i+1}, \dots \rangle \quad \text{in } \mathbb{R}[\lambda]. \quad (2.3)$$

It follows that  $V(g_3, \dots, g_{2\iota_p + 1}) = V(g_3, \dots, g_{2i+1}, \dots)$  and  $\max\{\zeta(\lambda'), \iota(\lambda')\} \leq \iota_p$ .

For  $\lambda' \in \mathbb{R}^m$  we need to discuss the sign changes among those focal values  $g_{2i+1}$  and the real part  $\alpha$  of the eigenvalues near  $(\alpha, \lambda) = (0, \lambda')$ . For convenience, define

$$g_1(\alpha) := 2\pi\alpha$$

complementarily. Our strategy is to restrict those  $g_{2i+1}$ 's ( $i = 0, 1, \dots, \kappa(\lambda')$ ), where  $\kappa(\lambda') = \zeta(\lambda')$  (or  $\iota(\lambda')$ ) if  $\lambda' \in \mathbb{R}^m \setminus \mathcal{C}$  (or  $\mathcal{C}$ ), to a curve in the space  $\mathbb{R}^{m+1}$  of parameters  $(\alpha, \lambda)$  to see independent sign changes in the displacement function. Consider a continuous curve  $\Upsilon$  of the form

$$(\alpha(\eta), \lambda(\eta)) := (d_0\eta^{\alpha_0}, \lambda'_1 + d_1\eta^{\alpha_1}, \dots, \lambda'_m + d_m\eta^{\alpha_m}) \quad (2.4)$$

in the parameter space  $\mathbb{R}^{m+1}$ , where  $d_i \in \mathbb{R}$  and  $\alpha_i > 0$  are indeterminate constants,  $i = 0, \dots, m$ , and  $(\lambda'_1, \dots, \lambda'_m) = \lambda'$ . Clearly,  $(\alpha(0), \lambda(0)) = (0, \lambda')$ , i.e., the curve passes through the point  $(0, \lambda')$ . The curve is of polynomial form if all  $\alpha_i$ 's are positive integers. Restricted to the curve  $\Upsilon$  given in (2.4), the focal values are of the form

$$g_1(\alpha(\eta)) = c_0\eta^{w_0} + o(\eta^{w_0}), \quad g_{2i+1}(\lambda(\eta)) = c_i\eta^{w_i} + o(\eta^{w_i}), \quad i = 1, \dots, \kappa(\lambda'), \quad (2.5)$$

where  $w_i$ 's are positive constants depending on the  $\alpha_i$ 's and the  $c_i$ 's are real constants depending on the  $d_i$ 's such that, for each  $i$ ,  $c_i \neq 0$  if and only if  $g_{2i+1}(\lambda(\eta)) \not\equiv 0$ . In particular,

$$c_{\kappa(\lambda')} = g_{2\kappa(\lambda') + 1}(\lambda') \neq 0 \quad \text{and} \quad w_{\kappa(\lambda')} = 0$$

if  $\lambda' \in \mathbb{R}^m \setminus \mathcal{C}$ .

Our method of restricted independence highly depends on the evolution in the power sequence  $w := \{w_i\}$ , where the  $w_i$ 's are reals unless that all  $\alpha_i$ 's are chosen as integers, because then the  $w_i$ 's are integers. It is worthy mentioning that the sequence  $\{w_i\}$  may not be increasing although the corresponding  $g_{2i+1}$  is given by the coefficient of the term  $\rho^{2i+1}$  in the return map (1.2). Let

$$\Delta(i, j)w := \frac{w_i - w_j}{i - j},$$

called the *difference quotient of  $w$  between  $i$  and  $j$* . For  $i_0 < \dots < i_k$  in  $\{0, 1, \dots, \kappa(\lambda')\}$ , the power sequence  $\{w_i\}$  is said to be *ladder-likely depressive* on the scale  $(i_0, \dots, i_k)$  if there are constants  $h_{i_0} > h_{i_1} > \dots > h_{i_k} > 0$ , called the *depressive rates*, such that

(LD) for each  $\nu = 0, \dots, k$ ,

$$\Delta(i_\nu, j)w \begin{cases} \leq -h_{i_\nu} & \forall j = 0, \dots, i_\nu - 1, \\ \geq -h_{i_\nu} & \forall j = i_\nu + 1, \dots, \kappa(\lambda'). \end{cases}$$

Obviously, the sequence  $\{7, 4, 2, 1\}$  is ladder-likely depressive on the scale  $(0, 1, 2, 3)$ , where we note that  $\{3, 2, 1\}$ , the sequence of differences between two consecutive terms, is strictly decreasing and we can choose the sequence  $\{6, 5/2, 3/2, 1/2\}$  for depressive rates. Note that the concept of ladder-like depressiveness does not require the sequence  $\{w_i\}$  to be decreasing but needs the existence of a decreasing subsequence of  $\{w_i\}$  with weaker and weaker depressive rates correspondingly. For example, the sequence  $\{7, 4, 2, 4, 1\}$  does not decrease but has a ladder-likely depressive scale  $(0, 1, 4)$  with the sequence  $(14, 5/2, 1/4)$  of depressive rates.

Considering the “=” in (LD), for each  $i_\nu$  define

$$\Xi(i_\nu) := \{i_\nu\} \cup \{j \in \{0, \dots, \kappa(\lambda')\} : \Delta(i_\nu, j)w = -h_{i_\nu}\},$$

the set of all  $j$ 's having the same slope  $-h_{i_\nu}$  with respect to  $i_\nu$ . Let

$$\begin{aligned} \hat{c}_{i_\nu} &:= \sum_{j \in \Xi(i_\nu)} c_j, \\ \mathcal{V} &:= \{i_\nu \in \{i_0, \dots, i_k\} : \exists j \in \{\nu + 1, \dots, k\} \text{ such that } \hat{c}_{i_\nu} \hat{c}_{i_j} < 0 \\ &\quad \text{and } \hat{c}_{i_l} = 0 \ \forall l = \nu + 1, \dots, j - 1\}, \end{aligned} \tag{2.6}$$

where  $c_i$  is the leading coefficient of  $g_{2i+1}$  as given in (2.5). Clearly  $\mathcal{V}$  is a set of indices for independent sign changes, i.e. an independence restricted to the parameterized curve  $\Upsilon$ .

**Theorem 2.1.** *Suppose that the power sequence of  $g_{2i+1}$ 's, the focal values of family  $LC(\alpha, \lambda)$  given in (1.1) near  $(0, \lambda')$ , restricted to the parameterized curve (2.4) is ladder-likely depressive on  $(i_0, \dots, i_k)$ , i.e., condition (LD) holds. Then  $\mathcal{N}(\lambda') \geq \#\mathcal{V}$ , the cardinality of the set  $\mathcal{V}$ .*

*Proof.* Let  $i_s$  be the greatest member of  $\mathcal{V}$  and  $s' \in \{s+1, \dots, k\}$  be the corresponding  $j$  given in the definition of  $\mathcal{V}$ . Clearly,

$$i_s < i_{s'} \quad \text{and} \quad \hat{c}_{i_s} \hat{c}_{i_{s'}} < 0.$$

From (1.2) we get

$$\begin{aligned} \Pi(\rho) &= g_1(\alpha)\rho(1 + \Psi_0(\alpha)) + \sum_{i=2}^{+\infty} g_i(\alpha, \boldsymbol{\lambda})\rho^i \\ &= \sum_{i=0}^{+\infty} g_{2i+1}\rho^{2i+1} (1 + \Psi_{2i+1}(\alpha, \boldsymbol{\lambda}, \rho)), \end{aligned} \quad (2.7)$$

where  $\Psi_0(\alpha)$  is an analytic function such that  $\Psi_0(0) = 0$ , and the functions  $\Psi_{2i+1}$ 's are analytic in  $(\alpha, \boldsymbol{\lambda}, \rho)$  and vanish at  $(0, \boldsymbol{\lambda}, 0)$ . On the other hand, by **(LD)** we get  $w_{i_\nu} - w_j \leq -(i_\nu - j)h_{i_\nu}$ , i.e.  $w_{i_\nu} + i_\nu h_{i_\nu} \leq w_j + j h_{i_\nu}$  for all  $j = 0, \dots, \kappa(\boldsymbol{\lambda}')$ . This implies that for each  $\nu \in \{0, \dots, k\}$  we have

$$w_{i_\nu} + (i_\nu + 1)h_{i_\nu} \leq w_j + (j + 1)h_{i_\nu} \quad \forall j = 0, \dots, \kappa(\boldsymbol{\lambda}'). \quad (2.8)$$

In what follows we use (2.7) and (2.8) to discuss in  $\mathbb{R}^m \setminus \mathcal{C}$  and  $\mathcal{C}$  separately.

For  $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$ , by (2.3) and (2.7) we get

$$\Pi(\rho) = \sum_{i=0}^{\iota_p} g_{2i+1}\rho^{2i+1} (1 + \Phi_{2i+1}(\alpha, \boldsymbol{\lambda}, \rho)), \quad (2.9)$$

where  $\Phi_{2i+1}$ 's are analytic at  $(\alpha, \boldsymbol{\lambda}, \rho)$  and vanish when  $\alpha = \rho = 0$ . Restricted to the curve  $(\alpha, \boldsymbol{\lambda}) = (\alpha(\eta), \boldsymbol{\lambda}(\eta))$ , from (2.9) we obtain

$$\begin{aligned} \Pi(\rho) &= \sum_{i=0}^{\zeta(\boldsymbol{\lambda}')} c_i \eta^{w_i} \rho^{2i+1} (1 + H_{2i+1}(\eta, \rho)) \\ &\quad + \begin{cases} \sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta)) \rho^{2i+1} (1 + \Phi_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta), \rho)) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \end{aligned}$$

where  $H_{2i+1}(\eta, \rho) \rightarrow 0$  as  $(\eta, \rho) \rightarrow (0, 0)$ . Then, for each  $\nu = 0, 1, \dots, k$ , we have

$$\begin{aligned} \Pi(\eta^{h_{i_\nu}/2}) &= \sum_{i=0}^{\zeta(\boldsymbol{\lambda}')} c_i \eta^{w_i + (i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\ &\quad + \begin{cases} \sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta)) \eta^{(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \end{aligned} \quad (2.10)$$

where

$$\tilde{H}_{2i+1}(\eta) := H_{2i+1}(\eta, \eta^{h_{i_\nu}/2}) \rightarrow 0 \quad \text{as } \eta \rightarrow 0. \quad (2.11)$$

Applying (2.8) to (2.10) we get

$$\begin{aligned}
\Pi(\eta^{h_{i_\nu}/2}) &= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} \sum_{i \in \Xi(i_\nu)} c_i \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\
&\quad + \sum_{i \in \{0, \dots, \zeta(\boldsymbol{\lambda}')\} \setminus \Xi(i_\nu)} c_i \eta^{w_i+(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\
&\quad + \begin{cases} \sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta)) \eta^{(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \\
&= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} \left\{ \hat{c}_{i_\nu} + \sum_{i \in \Xi(i_\nu)} c_i \tilde{H}_{2i+1}(\eta) \right\} \\
&\quad + \sum_{i \in \{0, \dots, \zeta(\boldsymbol{\lambda}')\} \setminus \Xi(i_\nu)} c_i \eta^{w_i+(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\
&\quad + \begin{cases} \sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta)) \eta^{(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \\
&= \hat{c}_{i_\nu} \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} (1 + \Psi_f(\eta)) \quad \forall i_\nu \in \mathcal{V} \cup \{i_{s'}\}, \tag{2.12}
\end{aligned}$$

where  $\hat{c}_{i_\nu}$  is defined in (2.6) and

$$\begin{aligned}
\Psi_f(\eta) &:= \sum_{i \in \Xi(i_\nu)} \frac{c_i}{\hat{c}_{i_\nu}} \tilde{H}_{2i+1}(\eta) + \sum_{i \in \{0, \dots, \zeta(\boldsymbol{\lambda}')\} \setminus \Xi(i_\nu)} \frac{c_i}{\hat{c}_{i_\nu}} \eta^{w_i-w_{i_\nu}+(i-i_\nu)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\
&\quad + \begin{cases} \sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} \frac{g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta))}{\hat{c}_{i_\nu}} \eta^{(i-i_\nu)h_{i_\nu}-w_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p. \end{cases} \tag{2.13}
\end{aligned}$$

By (2.11) we have that  $\Psi_f(\eta) \rightarrow 0$  as  $\eta \rightarrow 0$ .

For  $\boldsymbol{\lambda}' \in \mathcal{C}$ , from (2.2) and (2.7) we get

$$\Pi(\rho) = \sum_{i=0}^{\iota(\boldsymbol{\lambda}')} g_{2i+1} \rho^{2i+1} (1 + \Phi_{2i+1}(\alpha, \boldsymbol{\lambda}, \rho)), \tag{2.14}$$

where  $\Phi_{2i+1}$ 's are analytic at  $(\alpha, \boldsymbol{\lambda}', \rho)$  and vanish when  $\alpha = \rho = 0$ . Restricted to the curve  $(\alpha, \boldsymbol{\lambda}) = (\alpha(\eta), \boldsymbol{\lambda}(\eta))$ , from (2.14) we obtain

$$\Pi(\rho) = \sum_{i=0}^{\iota(\boldsymbol{\lambda}')} c_i \eta^{w_i} \rho^{2i+1} (1 + H_{2i+1}(\eta, \rho)),$$

where  $H_{2i+1}(\eta, \rho) \rightarrow 0$  as  $(\eta, \rho) \rightarrow (0, 0)$ . Then

$$\Pi(\eta^{h_{i_\nu}/2}) = \sum_{i=0}^{\iota(\boldsymbol{\lambda}')} c_i \eta^{w_i+(i+1/2)h_{i_\nu}} \left(1 + \tilde{H}_{2i+1}(\eta)\right) \quad \text{for } \nu = 0, 1, \dots, k, \tag{2.15}$$

where

$$\tilde{H}_{2i+1}(\eta) := H_{2i+1}(\eta, \eta^{h_{i_\nu}/2}) \rightarrow 0 \quad \text{as } \eta \rightarrow 0. \quad (2.16)$$

Applying (2.8) to (2.15) we get

$$\begin{aligned} \Pi(\eta^{h_{i_\nu}/2}) &= \eta^{w_{i_\nu} + (i_\nu + 1/2)h_{i_\nu}} \sum_{i \in \Xi(i_\nu)} c_i \left(1 + \tilde{H}_{2i+1}(\eta)\right) \\ &\quad + \sum_{i \in \{0, \dots, \iota(\mathbf{\lambda}')$$
\end{aligned}



### 3 Some corollaries

In this section we give two corollaries of Theorem 2.1 for easier applications in some cases. Those cases come from special cases of the condition **(LD)**.

Suppose that the power sequence of  $g_{2i+1}$ 's at  $(0, \boldsymbol{\lambda}')$  satisfies

$$(D) \quad w_0 - w_1 \geq \dots \geq w_{i-1} - w_i \geq \dots \geq w_{\kappa(\boldsymbol{\lambda}')-1} - w_{\kappa(\boldsymbol{\lambda}')} > 0.$$

This means that the sequence  $\{w_i\}$  is decreasing and the gaps between two consecutive terms become smaller and smaller.

For those “=” in condition **(D)**, we consider  $w_{\varsigma-1} - w_{\varsigma} = w_{\varsigma} - w_{\varsigma+1}$  for some  $\varsigma$  in  $\{0, 1, \dots, \kappa(\boldsymbol{\lambda}')\}$  and let integers  $\varsigma_1, \varsigma_2 \in \{0, \dots, \kappa(\boldsymbol{\lambda}')\}$  denote the indices such that  $\varsigma_1 < \varsigma < \varsigma_2$  and

$$w_{\varsigma_1-1} - w_{\varsigma_1} > w_{\varsigma_1} - w_{\varsigma_1+1} = \dots = w_{\varsigma_2-1} - w_{\varsigma_2} > w_{\varsigma_2} - w_{\varsigma_2+1}, \quad (3.1)$$

i.e.  $\varsigma_1$  and  $\varsigma_2$  are respectively the first left index and the first right index near  $\varsigma$  which destroy the equality “=” in **(D)**. Define  $w_{-1} = w_{\kappa(\boldsymbol{\lambda}')-1} := +\infty$  complementarily. Then, for each  $\varsigma \in \{0, 1, \dots, \kappa(\boldsymbol{\lambda}')\}$ , we define

$$\widehat{\Xi}(\varsigma) := \begin{cases} \{\varsigma_1, \dots, \varsigma, \dots, \varsigma_2\} & \text{if (3.1) holds,} \\ \{\varsigma\} & \text{if } w_{\varsigma-1} - w_{\varsigma} > w_{\varsigma} - w_{\varsigma+1}, \end{cases}$$

which can be used to find a scale  $(i_0, i_1, \dots, i_k)$ , where  $i_0 := 0$ ,  $i_k := \kappa(\boldsymbol{\lambda}')$  and  $i_1, \dots, i_{k-1} \in \{1, \dots, \kappa(\boldsymbol{\lambda}') - 1\}$  such that

$$\begin{aligned} \widehat{\Xi}(i_j) &\neq \widehat{\Xi}(i_l) && \text{for } 0 \leq j \neq l \leq k, \\ \widehat{\Xi}(i_j) &\neq \widehat{\Xi}(i_j - 1) && \text{for } 0 < j \leq k, \\ \cup_{\nu=0}^k \widehat{\Xi}(i_\nu) &= \{0, 1, \dots, \kappa(\boldsymbol{\lambda}')\}. \end{aligned}$$

On the scale we define the set  $\mathcal{V}$  as in (2.6), where

$$\hat{c}_{i_\nu} := \sum_{j \in \widehat{\Xi}(i_\nu)} c_j \quad \text{for } \nu = 0, 1, \dots, k,$$

and  $c_i$  is the leading coefficient of  $g_{2i+1}$  as given in (2.5).

**Corollary 3.1.**  $\mathcal{N}(\boldsymbol{\lambda}') \geq \#\mathcal{V}$  if the power sequence of  $g_{2i+1}$ 's at  $(0, \boldsymbol{\lambda}')$  satisfies condition **(D)**.

*Proof.* Under condition **(D)**, define

$$\begin{aligned} h_i &:= (w_{i-1} - w_{i+1})/2, && \forall i = 1, \dots, \kappa(\boldsymbol{\lambda}') - 1, \\ h_0 &:= 2(w_0 - w_1), && h_{\kappa(\boldsymbol{\lambda}')} := (w_{\kappa(\boldsymbol{\lambda}')-1} - w_{\kappa(\boldsymbol{\lambda}')})/2, \end{aligned} \quad (3.2)$$

i.e.  $h_i$  is the average of the differences  $w_{i-1} - w_i$  and  $w_i - w_{i+1}$ . From **(D)** and (3.2), it is easy to see that  $h_0 > h_i \geq h_j > h_{\kappa(\mathbf{\lambda}')} > 0$  for  $i < j$  in  $\{1, \dots, \kappa(\mathbf{\lambda}') - 1\}$ . Moreover,  $h_i = h_j$  if and only if  $w_{i-1} - w_i = w_i - w_{i+1} = \dots = w_j - w_{j+1}$ . By the choice of the scale  $(i_0, i_1, \dots, i_k)$  given before this corollary, we have that  $h_{i_0} > h_{i_1} > \dots > h_{i_k} > 0$ . We claim that this sequence  $\{w_i\}$  is ladder-likely degressive on the scale  $(i_0, i_1, \dots, i_k)$ . In fact, from **(D)** we see that for  $j < i_\nu$

$$\begin{aligned} (i_\nu - j)h_{i_\nu} &= (i_\nu - j) \left( \frac{w_{i_\nu-1} - w_{i_\nu}}{2} + \frac{w_{i_\nu} - w_{i_\nu+1}}{2} \right) \leq (i_\nu - j)(w_{i_\nu-1} - w_{i_\nu}) \\ &\leq \sum_{i=j+1}^{i=i_\nu} (w_{i-1} - w_i) = w_j - w_{i_\nu}, \end{aligned}$$

where “=” holds if and only if  $w_j - w_{j+1} = \dots = w_{i_\nu-1} - w_{i_\nu} = w_{i_\nu} - w_{i_\nu+1}$  because of **(D)**. Similarly, for  $j > i_\nu$  we obtain

$$\begin{aligned} (i_\nu - j)h_{i_\nu} &= (i_\nu - j) \left( \frac{w_{i_\nu-1} - w_{i_\nu}}{2} + \frac{w_{i_\nu} - w_{i_\nu+1}}{2} \right) \leq (i_\nu - j)(w_{i_\nu} - w_{i_\nu+1}) \\ &= (j - i_\nu)(w_{i_\nu+1} - w_{i_\nu}) \leq \sum_{i=i_\nu+1}^{i=j} (w_i - w_{i-1}) = w_j - w_{i_\nu}, \end{aligned}$$

where “=” holds if and only if  $w_{i_\nu-1} - w_{i_\nu} = w_{i_\nu} - w_{i_\nu+1} = \dots = w_{j-1} - w_j$ . Thus, (2.8) holds in our case, which implies that the  $h_{i_\nu}$ 's ( $\nu = 0, 1, \dots, k$ ) satisfy condition **(LD)** and therefore the claim is proved. Therefore,  $\mathcal{N}(\mathbf{\lambda}') \geq \#\mathcal{V}$  by Theorem 2.1.  $\square$

Since condition **(D)** requires that the sequence of power-differences  $\{w_0 - w_1, \dots, w_{\kappa(\mathbf{\lambda}')-1} - w_{\kappa(\mathbf{\lambda}')}\}$  is non-increasing, we have the following three cases:

**(D0)** there is no “=” in **(D)**;

**(D1)** “=” appears in **(D)** in discontinuous manner;

**(D2)** “=” appears continuously in **(D)**, i.e. there exists  $i$  such that  $\dots \geq w_{i-1} - w_i = w_i - w_{i+1} = w_{i+1} - w_{i+2} \geq \dots$

**Corollary 3.2.**  $\mathcal{N}(\mathbf{\lambda}') = \kappa(\mathbf{\lambda}')$  if either  $c_i c_{i+1} < 0$  for all  $i = 0, \dots, \kappa(\mathbf{\lambda}') - 1$  in the case **(D0)**, or  $\hat{c}_{i_\nu} \hat{c}_{i_\nu+1} < 0$  for all  $\nu = 0, \dots, \kappa(\mathbf{\lambda}') - 1$  in the case **(D1)**.

*Proof.* In case **(D0)** choosing the scale  $(i_0, i_1, \dots, i_k)$  as  $(0, 1, \dots, \kappa(\mathbf{\lambda}'))$  and defining the  $h_{i_\nu}$ 's as in (3.2), we compute  $\hat{c}_{i_\nu} = c_{i_\nu}$  for all  $\nu = 0, 1, \dots, \kappa(\mathbf{\lambda}')$ . In case **(D1)** choosing the scale  $(i_0, i_1, \dots, i_k)$  and defining the  $h_{i_\nu}$ 's as above, we see that  $k = \kappa(\mathbf{\lambda}')$  and  $h_{i_0} > h_{i_1} > \dots > h_{i_k} > 0$ . Then the result follows from Corollary 3.1.  $\square$

Although the cyclicity  $\mathcal{N}(\mathbf{\lambda}')$  may reach the upper estimate  $\kappa(\mathbf{\lambda}')$  in cases **(D0)** and **(D1)**, for which Corollary 3.2 gives sufficient conditions, we do not have such a

result yet in case **(D2)** because the equality  $\widehat{\Xi}(i) = \widehat{\Xi}(i+1)$  known by the definition of  $\widehat{\Xi}$  implies that  $\#\mathcal{V} \leq k < \kappa(\boldsymbol{\lambda}')$ .

Remark that Theorem 6.6 of [11] can also be employed to case **(D0)** but does not work for case **(D1)**. On the other hand,  $\hat{c}_{i_\nu} = c_{i_\nu}$  for all  $\nu = 0, 1, \dots, \kappa(\boldsymbol{\lambda}')$  in case **(D0)** but, there are some  $i$  such that  $\hat{c}_{i_\nu} = c_{i_{\nu-1}} + c_{i_\nu} + c_{i_{\nu+1}}$  in case **(D1)**. For example,  $\hat{c}_0 = c_0$ ,  $\hat{c}_1 = c_1$ ,  $\hat{c}_2 = c_1 + c_2 + c_3$  and  $\hat{c}_3 = c_3$  when  $w_0 - w_1 > w_1 - w_2 = w_2 - w_3 > w_3 - w_4 \dots$ .

We consider the following family of polynomial differential systems

$$\dot{x} = \alpha x - y + \sum_{i=1}^4 a_{2i+1} x(x^2 + y^2)^i, \quad \dot{y} = x + \alpha y + \sum_{i=1}^4 a_{2i+1} y(x^2 + y^2)^i, \quad (3.3)$$

parameterized by  $(\alpha, \boldsymbol{\lambda}) := (\alpha, \lambda_1, \lambda_2) \in \mathbb{R}^3$ , where  $a_3 := \lambda_1$ ,  $a_5 := -\lambda_2^2$ ,  $a_7 := 3\lambda_2$  and  $a_9 := -1$ . One can compute its focal values

$$g_1 = 2\pi\alpha, \quad g_3 = 2\pi\lambda_1, \quad g_5 = -2\pi\lambda_2^2, \quad g_7 = 6\pi\lambda_2, \quad g_9 = -2\pi, \quad (3.4)$$

where each  $g_{2i+1}$  is the remainder of the original  $g_{2i+1}$  divided by the Gröbner basis of ideal  $\langle g_3, \dots, g_{2i-1} \rangle$  in the order  $\lambda_1 \prec \lambda_2$ . Thus,  $\kappa(\boldsymbol{\lambda}') = \zeta(\boldsymbol{\lambda}') = 4$ , where  $\boldsymbol{\lambda}' = (0, 0)$ . Note that the independence condition of focal values, i.e. **(ID<sub>k</sub>-1)** and **(ID<sub>k</sub>-2)**, do not hold for  $g_1, g_3, \dots, g_7$  because  $g_7 = 0$  if  $g_5 = 0$ , which implies that we cannot obtain 4 limit cycles by verifying the classical independence of focal values. Using our above mentioned method, we choose the curve

$$\Upsilon : \alpha = -\eta^9, \quad \lambda_1 = \eta^5, \quad \lambda_2 = \eta,$$

in the  $(\alpha, \lambda_1, \lambda_2)$ -space. Restricted to  $\Upsilon$  those focal values given in (3.4) can be written in the form (2.5) taking

$$\begin{aligned} w_0 &= 9, & w_1 &= 5, & w_2 &= 2, & w_3 &= 1, & w_4 &= 0, \\ c_0 &= -2\pi, & c_1 &= 2\pi, & c_2 &= -2\pi, & c_3 &= 6\pi, & c_4 &= -2\pi. \end{aligned}$$

One can check that

$$w_0 - w_1 > w_1 - w_2 > w_2 - w_3 = w_3 > 0,$$

and compute that

$$\hat{c}_0 = c_0 = -2\pi, \quad \hat{c}_1 = c_1 = 2\pi, \quad \hat{c}_2 = c_2 = -2\pi, \quad \hat{c}_3 = c_2 + c_3 + c_4 = 2\pi, \quad \hat{c}_4 = c_4 = -2\pi,$$

which implies by Corollary 3.2 in case **(D1)** that  $\mathcal{N}(\boldsymbol{\lambda}') = 4$ .

## 4 Application to cubic systems

In this section we apply our method to a family of cubic polynomial differential systems with 5 parameters. Consider

$$\begin{aligned} \dot{x} &= \alpha x - y + (\lambda_1 - \lambda_3)x^2 + \lambda_2 xy + \lambda_3 y^2 - (9 + \lambda_2^2 + \lambda_3 \lambda_4)x^2 y + 2y^3, \\ \dot{y} &= x + \alpha y - x^3 - (12 + \lambda_2^2 + \lambda_3^2 + \lambda_3 \lambda_4)xy^2, \end{aligned} \quad (4.1)$$

parameterized by  $(\alpha, \boldsymbol{\lambda}) := (\alpha, \lambda_1, \dots, \lambda_4) \in \mathbb{R}^5$ . It is easy to compute the first five nonzero focal values

$$g_1 = 2\pi\alpha, \quad g_3 = \frac{\pi}{4}\lambda_1\lambda_2, \quad g_5 = -\frac{\pi}{24}\lambda_2\lambda_3^3, \quad g_7 = \frac{\pi}{96}\lambda_2\lambda_3^2\lambda_4, \quad g_9 = -\frac{\Omega(\lambda_2)\pi}{960}\lambda_2\lambda_3, \quad (4.2)$$

where  $\Omega(\lambda) := 6720 + 1265\lambda^2 + 61\lambda^4$  and, for a short statement, each  $g_{2i+1}$  is the remainder of the original  $g_{2i+1}$  divided by the Gröbner basis of ideal  $\langle g_3, \dots, g_{2i-1} \rangle$  in the order  $\lambda_1 \prec \lambda_2 \prec \lambda_3 \prec \lambda_4$ . Family (4.1) has the center variety  $\mathcal{C} = \Gamma_1 \cup \Gamma_2$ , where

$$\Gamma_1 := \{\boldsymbol{\lambda} \in \mathbb{R}^4 : \lambda_1 = \lambda_3 = 0\} \quad \text{and} \quad \Gamma_2 := \{\boldsymbol{\lambda} \in \mathbb{R}^4 : \lambda_2 = 0\}.$$

In fact, family (4.1)| $_{\alpha=0}$  is time-reversible for  $\boldsymbol{\lambda} \in \Gamma_1 \cup \Gamma_2$ , but on the contrary

$$V(g_3) \cap \mathbb{R}^4 \supsetneq V(g_3, g_5) \cap \mathbb{R}^4 = \Gamma_1 \cup \Gamma_2 = V(g_3, \dots, g_{2i+1}, \dots) \cap \mathbb{R}^4, \quad (4.3)$$

by the expressions of  $g_3$  and  $g_5$ .

**Proposition 4.1.** *For  $\boldsymbol{\lambda} \in \mathbb{R}^4 \setminus \mathcal{C}$  the cyclicity  $\mathcal{N}_f(\boldsymbol{\lambda})$  and the multiplicity  $\zeta(\boldsymbol{\lambda})$  of  $O$  in the family (4.1) satisfy that either  $\mathcal{N}_f(\boldsymbol{\lambda}) = \zeta(\boldsymbol{\lambda}) = 1$  or  $\mathcal{N}_f(\boldsymbol{\lambda}) = \zeta(\boldsymbol{\lambda}) = 2$ , which holds if either  $\lambda_1\lambda_2 \neq 0$  or  $\lambda_1 = 0 \neq \lambda_2\lambda_3$  correspondingly.*

*Proof.* The results can be proved by checking the independence condition of focal values, i.e. **(ID<sub>k</sub>-1)** and **(ID<sub>k</sub>-2)** for  $k = 2$ . Actually, by (4.3), the origin is a weak focus of multiplicity at most 2 when  $\boldsymbol{\lambda} \in \mathbb{R}^4 \setminus \mathcal{C}$ . For such a  $\boldsymbol{\lambda}$ , by the definitions of  $\Gamma_1$  and  $\Gamma_2$ , there are only two cases: either  $\lambda_1\lambda_2 \neq 0$  or  $\lambda_1 = 0 \neq \lambda_2\lambda_3$ . It is easy to check that  $g_1, g_3$  and  $g_5$  are independent at  $(0, \boldsymbol{\lambda})$  when  $\boldsymbol{\lambda}$  satisfies that  $\lambda_1 = 0 \neq \lambda_2\lambda_3$ , and that  $g_1$  and  $g_3$  are independent at  $(0, \boldsymbol{\lambda})$  when  $\boldsymbol{\lambda}$  satisfies that  $\lambda_1\lambda_2 \neq 0$ .

Meanwhile, the results of this proposition can also be proved by using our main theorem or corollaries. In fact, in the case that  $\lambda_1 = 0 \neq \lambda_2\lambda_3$ , consider the parametric curve

$$\alpha(\eta) := -\text{sgn}(\lambda_2\lambda_3)\eta^{10}, \quad \lambda_1(\eta) := \text{sgn}(\lambda_3)\eta^3, \quad \lambda_i(\eta) := \lambda_i + \eta, \quad i = 2, 3, 4.$$

With the restriction to the curve, we can compute  $w_0 = 10, w_1 = 3, w_2 = 0$  and  $c_0 = -2\pi\text{sgn}(\lambda_2\lambda_3), c_1 = \pi\lambda_2\text{sgn}(\lambda_3)/4, c_2 = -\pi\lambda_2\lambda_3^3/24$ . By Corollary 3.2,  $\mathcal{N}_f(\boldsymbol{\lambda}) = 2$ .

In the case that  $\lambda_1\lambda_2 \neq 0$ , we can prove  $\mathcal{N}_f(\boldsymbol{\lambda}) = 1$  similarly.  $\square$

This proposition shows that, for  $\boldsymbol{\lambda} \notin \Gamma_1 \cup \Gamma_2$ , the origin  $O$  is a weak focus of multiplicity at most 2, and there are small perturbations such that exactly  $j$  limit cycles bifurcate from the weak focus of multiplicity  $j$  for  $j = 1, 2$ .

On the other hand, the origin  $O$  is a center of (4.1) if and only if  $\alpha = 0$  and  $\boldsymbol{\lambda} \in \Gamma_1 \cup \Gamma_2$ . Clearly, every point in  $\Gamma_1$  and  $\Gamma_2$  can be written as  $\boldsymbol{\lambda}^{(1)} := (0, \lambda_2^{(1)}, 0, \lambda_4^{(1)})$  and  $\boldsymbol{\lambda}^{(2)} := (\lambda_1^{(2)}, 0, \lambda_3^{(2)}, \lambda_4^{(2)})$  respectively. In order to avoid a double discussion at the intersection of  $\Gamma_1 \cap \Gamma_2$ , we assume either  $\lambda_1^{(2)} \neq 0$  or  $\lambda_3^{(2)} \neq 0$ .

**Proposition 4.2.** *For  $\lambda$  equal to  $\lambda^{(1)}$  or  $\lambda^{(2)}$  in  $\mathcal{C}$ , the cyclicity  $\mathcal{N}_c(\lambda)$  and the multiplicity  $\iota(\lambda)$  of the origin  $O$  in the family (4.1) have the results given in Table 1:*

For	if	then
$\lambda^{(1)} := (0, \lambda_2^{(1)}, 0, \lambda_4^{(1)})$	$\lambda_4^{(1)} \neq 0, \vartheta > 0$	$\iota(\lambda^{(1)}) = 4, \mathcal{N}_c(\lambda^{(1)}) = 4$
	$\lambda_4^{(1)} \neq 0, \vartheta \leq 0$	$\iota(\lambda^{(1)}) = 4, \mathcal{N}_c(\lambda^{(1)}) \geq 2$
	$\lambda_4^{(1)} = 0$	$\iota(\lambda^{(1)}) = 4, \mathcal{N}_c(\lambda^{(1)}) \geq 2$
$\lambda^{(2)} := (\lambda_1^{(2)}, 0, \lambda_3^{(2)}, \lambda_4^{(2)})$ where either $\lambda_1^{(2)} \neq 0$ or $\lambda_3^{(2)} \neq 0$	$\lambda_1^{(2)} = 0, \lambda_3^{(2)} \neq 0$	$\iota(\lambda^{(2)}) = 2, \mathcal{N}_c(\lambda^{(2)}) = 2$
	$\lambda_1^{(2)} \neq 0$	$\iota(\lambda^{(2)}) = 1, \mathcal{N}_c(\lambda^{(2)}) = 1$
Remark: $\vartheta := 5(\lambda_4^{(1)})^2 - 8\Omega(\lambda_2^{(1)})$ and $\Omega$ is given in (4.2).		

Table 1: The number of limit cycles bifurcating from the center  $O$ .

*Proof.* By the definition of  $\Gamma_1$  and  $\Gamma_2$  every focal value of family (4.1) is of the form

$$\lambda_2(f_1(\lambda)\lambda_1 + f_2(\lambda)\lambda_3), \quad (4.4)$$

where  $f_1, f_2 \in \mathbb{R}[\lambda]$ . From (4.4) we see for  $\lambda = \lambda^{(1)}$  that the least integer  $j$  such that

$$\langle g_3, \dots, g_{2j+1} \rangle_{\lambda^{(1)}} = \langle g_3, \dots, g_{2i+1}, \dots \rangle_{\lambda^{(1)}}$$

in  $\mathbb{R}\{\lambda\}_{\lambda^{(1)}}$  is 4, because every focal value lies in  $\langle \lambda_1\lambda_2, \lambda_2\lambda_3 \rangle_{\lambda^{(1)}}$ ,  $\lambda_1\lambda_2, \lambda_2\lambda_3 \in \langle g_3, g_5, g_7, g_9 \rangle_{\lambda^{(1)}}$  and  $\lambda_2\lambda_3 \notin \langle g_3, g_5, g_7 \rangle_{\lambda^{(1)}}$ . This implies  $\iota(\lambda^{(1)}) = 4$ . For  $\lambda = \lambda^{(2)}$  we consider the case that  $\lambda_1^{(2)} = 0$  but  $\lambda_3^{(2)} \neq 0$ , and the case that  $\lambda_1^{(2)} \neq 0$  separately. In the first case  $\langle g_3 \rangle_{\lambda^{(2)}} \neq \langle g_3, \dots, g_{2i+1}, \dots \rangle_{\lambda^{(2)}}$  and  $\langle g_3, g_5 \rangle_{\lambda^{(2)}} = \langle g_3, \dots, g_{2i+1}, \dots \rangle_{\lambda^{(2)}}$  in  $\mathbb{R}\{\lambda\}_{\lambda^{(2)}}$ , which implies that  $\iota(\lambda^{(2)}) = 2$ . In the second case  $\lambda_2^{(2)} \in \langle g_3 \rangle_{\lambda^{(2)}}$  in  $\mathbb{R}\{\lambda\}_{\lambda^{(2)}}$  by the expression of  $g_3$ . Thus,  $\langle g_3 \rangle_{\lambda^{(2)}} = \langle g_3, \dots, g_{2i+1}, \dots \rangle_{\lambda^{(2)}}$  in  $\mathbb{R}\{\lambda\}_{\lambda^{(2)}}$ , which implies  $\iota(\lambda^{(2)}) = 1$ .

First, consider  $\lambda^{(1)}$  in the case that  $\lambda_4^{(1)} \neq 0$ . Let

$$\begin{aligned} \alpha(\eta) &:= 0 + d_0\eta^{\alpha_0}, & \lambda_1(\eta) &:= 0 + d_1\eta^{\alpha_1}, & \lambda_2(\eta) &:= \lambda_2^{(1)} + d_2\eta^{\alpha_2}, \\ & & \lambda_3(\eta) &:= 0 + d_3\eta^{\alpha_3}, & \lambda_4(\eta) &:= \lambda_4^{(1)} + d_4\eta^{\alpha_4}, \end{aligned} \quad (4.5)$$

where the  $\alpha_j$ 's and the  $d_j$ 's are undetermined. Then, we obtain

$$\begin{aligned} g_1(\alpha(\eta)) &= 2\pi d_0\eta^{\alpha_0}, \\ g_3(\lambda(\eta)) &= \left( \pi d_1\lambda_2^{(1)} + \pi d_1d_2\eta^{\alpha_2} \right) \eta^{\alpha_1}/4, \\ g_5(\lambda(\eta)) &= - \left( \pi\lambda_2^{(1)}d_3^3 + \pi d_2d_3^3\eta^{\alpha_2} \right) \eta^{3\alpha_3}/24, \\ g_7(\lambda(\eta)) &= \left( \pi\lambda_2^{(1)}\lambda_4^{(1)}d_3^2 + \pi\lambda_2^{(1)}d_3^2d_4\eta^{\alpha_4} + \pi d_2d_3^2\lambda_4^{(1)}\eta^{\alpha_2} + \pi d_2d_3^2d_4\eta^{\alpha_2+\alpha_4} \right) \eta^{2\alpha_3}/96, \\ g_9(\lambda(\eta)) &= - \left( \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi\lambda_2^{(1)}d_3 + \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi d_2d_3\eta^{\alpha_2} \right) \eta^{\alpha_3}/960, \end{aligned}$$

which give the power sequence  $\{w_i\}$  and the leading coefficients  $c_i$ 's as follows:

$$\begin{aligned} w_0 &= \alpha_0, & w_1 &= \alpha_1, & w_2 &= 3\alpha_3, & w_3 &= 2\alpha_3, & w_4 &= \alpha_3, \\ c_0 &= 2\pi d_0, & c_1 &= \frac{\pi}{4}d_1\lambda_2^{(1)}, & c_2 &= -\frac{\pi}{24}d_3^3\lambda_2^{(1)}, & c_3 &= \frac{\pi}{96}d_3^2\lambda_2^{(1)}\lambda_4^{(1)}, & c_4 &= -\Omega(\lambda_2^{(1)})\frac{\pi}{960}d_3\lambda_2^{(1)}, \end{aligned}$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$\begin{aligned} w_0 &= \alpha_0, & w_1 &= \alpha_1 + \alpha_2, & w_2 &= 3\alpha_3 + \alpha_2, & w_3 &= 2\alpha_3 + \alpha_2, & w_4 &= \alpha_3 + \alpha_2, \\ c_0 &= 2\pi d_0, & c_1 &= \frac{\pi}{4}d_1d_2, & c_2 &= -\frac{\pi}{24}d_2d_3^3, & c_3 &= \frac{\pi}{96}d_2d_3^2\lambda_4^{(1)}, & c_4 &= -\Omega(0)\frac{\pi}{960}d_2d_3, \end{aligned}$$

when  $\lambda_2^{(1)} = 0$ . We claim that there exist positive numbers  $\alpha_i$ 's in (4.5) such that

$$w_0 - w_1 > w_1 - w_2 > w_2 - w_3 = w_3 - w_4 > 0. \quad (4.6)$$

In fact, (4.6) is equivalent to either  $\alpha_0 - \alpha_1 > \alpha_1 - 3\alpha_3 > \alpha_3$  when  $\lambda_2^{(1)} \neq 0$  or  $\alpha_0 - \alpha_1 - \alpha_2 > \alpha_1 - 3\alpha_3 > \alpha_3$  when  $\lambda_2^{(1)} = 0$ , from which we can choose

$$\alpha_0 = 100, \quad \alpha_1 = 50, \quad \alpha_2 = 10, \quad \alpha_3 = 10,$$

for both cases. Thus, by (4.6) our system falls into the case **(D1)**. From the definition (2.6) we compute

$$\begin{aligned} \hat{c}_0 &= c_0 = 2\pi d_0, & \hat{c}_1 &= c_1 = \pi d_1\lambda_2^{(1)}/4, & \hat{c}_2 &= c_2 = -\pi d_3^3\lambda_2^{(1)}/24, \\ \hat{c}_3 &= c_2 + c_3 + c_4 = -\frac{\pi}{960}d_3\lambda_2^{(1)}\left(40d_3^2 - 10\lambda_4^{(1)}d_3 + \Omega(\lambda_2^{(1)})\right), & \hat{c}_4 &= c_4 = -\Omega(\lambda_2^{(1)})\frac{\pi}{960}d_3\lambda_2^{(1)}, \end{aligned}$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$\begin{aligned} \hat{c}_0 &= c_0 = 2\pi d_0, & \hat{c}_1 &= c_1 = \pi d_1d_2/4, & \hat{c}_2 &= c_2 = -\pi d_2d_3^3/24, \\ \hat{c}_3 &= c_2 + c_3 + c_4 = -\pi d_2d_3\left(40d_3^2 - 10\lambda_4^{(1)}d_3 + \Omega(0)\right)/960, & \hat{c}_4 &= c_4 = -\Omega(0)\pi d_2d_3/960, \end{aligned}$$

when  $\lambda_2^{(1)} = 0$ . Then, if  $\vartheta > 0$  we can check that

$$\hat{c}_0\hat{c}_1 < 0, \quad \hat{c}_1\hat{c}_2 < 0, \quad \hat{c}_2\hat{c}_3 < 0, \quad \hat{c}_3\hat{c}_4 < 0,$$

where we choose in (4.5) either

$$d_0 = -\text{sign}(\lambda_2^{(1)}\lambda_4^{(1)}), \quad d_1 = \text{sign}(\lambda_4^{(1)}), \quad d_2 = 0, \quad d_3 = \lambda_4^{(1)}/8, \quad d_4 = 0$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$d_0 = -\text{sign}(\lambda_4^{(1)}), \quad d_1 = \text{sign}(\lambda_4^{(1)}), \quad d_2 = 1, \quad d_3 = \lambda_4^{(1)}/8, \quad d_4 = 0$$

when  $\lambda_2^{(1)} = 0$ . Therefore  $\mathcal{N}_c(\mathbf{\lambda}^{(1)}) = 4$  by Corollary 3.2. Similarly, if  $\vartheta \leq 0$  we can check that

$$\hat{c}_0\hat{c}_1 < 0, \quad \hat{c}_1\hat{c}_2 < 0,$$

where we choose in (4.5) either

$$d_0 = -\text{sign}(\lambda_2^{(1)}), \quad d_1 = 1, \quad d_2 = 0, \quad d_3 = 1, \quad d_4 = 0$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$d_0 = -1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 0$$

when  $\lambda_2^{(1)} = 0$ . Therefore  $\mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$  by Corollary 3.1.

Next, consider  $\boldsymbol{\lambda}^{(1)}$  in the case that  $\lambda_4^{(1)} = 0$ . Let

$$\begin{aligned} \alpha(\eta) &:= 0 + d_0\eta^{\alpha_0}, \quad \lambda_1(\eta) := 0 + d_1\eta^{\alpha_1}, \quad \lambda_2(\eta) := \lambda_2^{(1)} + d_2\eta^{\alpha_2}, \\ \lambda_3(\eta) &:= 0 + d_3\eta^{\alpha_3}, \quad \lambda_4(\eta) := 0 + d_4\eta^{\alpha_4}. \end{aligned} \quad (4.7)$$

Then

$$\begin{aligned} g_1(\alpha(\eta)) &= 2\pi d_0\eta^{\alpha_0}, \\ g_3(\boldsymbol{\lambda}(\eta)) &= \left( \pi d_1 \lambda_2^{(1)} \eta^{\alpha_1} + \pi d_1 d_2 \eta^{\alpha_1 + \alpha_2} \right) / 4, \\ g_5(\boldsymbol{\lambda}(\eta)) &= - \left( \pi \lambda_2^{(1)} d_3^3 \eta^{3\alpha_3} + \pi d_2 d_3^3 \eta^{3\alpha_3 + \alpha_2} \right) / 24, \\ g_7(\boldsymbol{\lambda}(\eta)) &= \left( \pi \lambda_2^{(1)} d_3^2 d_4 \eta^{2\alpha_3 + \alpha_4} + \pi d_2 d_3^2 d_4 \eta^{2\alpha_3 + \alpha_2 + \alpha_4} \right) / 96, \\ g_9(\boldsymbol{\lambda}(\eta)) &= - \left( \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2}) \pi \lambda_2^{(1)} d_3 \eta^{\alpha_3} + \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2}) \pi d_2 d_3 \eta^{\alpha_2 + \alpha_3} \right) / 960, \end{aligned}$$

which gives the power sequence  $\{w_i\}$  and the leading coefficients  $c_i$ 's as follows:

$$\begin{aligned} w_0 &= \alpha_0, \quad w_1 = \alpha_1, \quad w_2 = 3\alpha_3, \quad w_3 = 2\alpha_3 + \alpha_4, \quad w_4 = \alpha_3, \\ c_0 &= 2\pi d_0, \quad c_1 = \frac{\pi}{4} d_1 \lambda_2^{(1)}, \quad c_2 = -\frac{\pi}{24} d_3^3 \lambda_2^{(1)}, \quad c_3 = \frac{\pi}{96} d_3^2 d_4 \lambda_2^{(1)}, \quad c_4 = -\Omega(\lambda_2^{(1)}) \frac{\pi}{960} d_3 \lambda_2^{(1)}, \end{aligned}$$

when  $\lambda_2^{(1)} \neq 0$  and

$$\begin{aligned} w_0 &= \alpha_0, \quad w_1 = \alpha_1 + \alpha_2, \quad w_2 = 3\alpha_3 + \alpha_2, \quad w_3 = 2\alpha_3 + \alpha_2 + \alpha_4, \quad w_4 = \alpha_3 + \alpha_2, \\ c_0 &= 2\pi d_0, \quad c_1 = \frac{\pi}{4} d_1 d_2, \quad c_2 = -\frac{\pi}{24} d_2 d_3^3, \quad c_3 = \frac{\pi}{96} d_2 d_3^2 d_4, \quad c_4 = -\Omega(0) \frac{\pi}{960} d_2 d_3, \end{aligned}$$

when  $\lambda_2^{(1)} = 0$ . We claim that there exist positive numbers  $\alpha_i$ 's in (4.7) such that

$$w_0 - w_1 > w_1 - w_2 > w_2 - w_3 < w_3 - w_4. \quad (4.8)$$

In fact, (4.8) is equivalent to either  $\alpha_0 - \alpha_1 > \alpha_1 - 3\alpha_3 > \alpha_3 - \alpha_4 < \alpha_3 + \alpha_4$  when  $\lambda_2^{(1)} \neq 0$ , or  $\alpha_0 - \alpha_1 - \alpha_2 > \alpha_1 - 3\alpha_3 > \alpha_3 - \alpha_4 < \alpha_3 + \alpha_4$  when  $\lambda_2^{(1)} = 0$ , from which we can choose

$$\alpha_0 = 13, \quad \alpha_1 = 8, \quad \alpha_2 = 3, \quad \alpha_3 = 2, \quad \alpha_4 = 1,$$

when  $\lambda_2^{(1)} \neq 0$ , and

$$\alpha_0 = 18, \quad \alpha_1 = 10, \quad \alpha_2 = 3, \quad \alpha_3 = 2, \quad \alpha_4 = 1,$$

when  $\lambda_2^{(1)} = 0$ . Thus **(LD)** holds on the scale  $(i_0, i_1, i_2) := (0, 1, 2)$ , where

$$h_{i_0} := 10, \quad h_{i_1} := 3.5, \quad h_{i_2} := 1.5,$$

when  $\lambda_2^{(1)} \neq 0$ , and

$$h_{i_0} := 10, \quad h_{i_1} := 4.5, \quad h_{i_2} := 2.5,$$

when  $\lambda_2^{(1)} = 0$ . Moreover “=” in **(LD)** holds only for  $j = i_\nu$  ( $\nu = 0, 1, 2$ ). From the definition of  $\hat{c}_{i_\nu}$  we compute

$$\hat{c}_{i_0} = c_0 = 2\pi d_0, \quad \hat{c}_{i_1} = c_1 = \pi d_1 \lambda_2^{(1)} / 4, \quad \hat{c}_{i_2} = c_2 = -\pi d_3^3 \lambda_2^{(1)} / 24,$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$\hat{c}_{i_0} = c_0 = 2\pi d_0, \quad \hat{c}_{i_1} = c_1 = \pi d_1 d_2 / 4, \quad \hat{c}_{i_2} = c_2 = -\pi d_2 d_3^3 / 24,$$

when  $\lambda_2^{(1)} = 0$ . We can check that

$$\hat{c}_{i_0} \hat{c}_{i_1} < 0, \quad \hat{c}_{i_1} \hat{c}_{i_2} < 0,$$

where we choose in (4.7) either

$$d_0 = -\text{sign}(\lambda_2^{(1)}), \quad d_1 = d_2 = d_3 = d_4 = 1,$$

when  $\lambda_2^{(1)} \neq 0$ , or

$$d_0 = -1, \quad d_1 = d_2 = d_3 = d_4 = 1,$$

when  $\lambda_2^{(1)} = 0$ . Therefore  $\mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$  by Theorem 2.1.

We similarly consider  $\boldsymbol{\lambda}^{(2)}$  and obtain  $\mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 2$  in the case that  $\lambda_1^{(2)} = 0, \lambda_3^{(2)} \neq 0$  and  $\mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 1$  in the case that  $\lambda_1^{(2)} \neq 0$ .  $\square$

Proposition 4.2 implies that for some  $\boldsymbol{\lambda}' \in \mathcal{C}$  there are 4 limit cycles bifurcating from the center  $O$  of system (4.1) $|_{(\alpha, \boldsymbol{\lambda})=(0, \boldsymbol{\lambda}')}$ . However, it is impossible to obtain 4 limit cycles bifurcating from the center by either the well-known method of independent focal values, or the method given in [11, Theorem 6.6]. In fact, the focal values  $g_3, g_5, g_7, g_9$  are not independent because  $g_7 = 0$  when  $g_5 = 0$ . We can compute

$$\frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, \dots, \lambda_4)} = \begin{pmatrix} \frac{\pi}{4} \lambda_2 & \frac{\pi}{4} \lambda_1 & 0 & 0 \\ 0 & -\frac{\pi}{24} \lambda_3^3 & -\frac{\pi}{8} \lambda_2 \lambda_3^2 & 0 \\ 0 & \frac{\pi}{96} \lambda_3^2 \lambda_4 & \frac{\pi}{48} \lambda_2 \lambda_3 \lambda_4 & \frac{\pi}{96} \lambda_2 \lambda_3^2 \\ 0 & -\frac{F(\lambda_2)\pi}{960} \lambda_3 & -\frac{G(\lambda_2)\pi}{960} \lambda_2 & 0 \end{pmatrix},$$

where  $F(\lambda) := 6720 + 3795\lambda^2 + 305\lambda^4$  and  $G(\lambda) := 6720 + 1265\lambda^2 + 61\lambda^4$ , and obtain

$$\text{rank} \left( \frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, \dots, \lambda_4)} \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(1)}} \right) \leq 2, \quad \text{rank} \left( \frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, \dots, \lambda_4)} \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(2)}} \right) \leq 1,$$

which implies by [3, Theorem 1.3] that 2 limit cycles can be bifurcated from the center  $O$  for the case that  $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(1)}$ , and 1 limit cycle for the case that  $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(2)}$ . Moreover, from the expressions of the  $g_{2i+1}$ 's given in (4.2) we find that “=” always appears in **(D)**, which implies that the result of [11, Theorem 6.6] cannot be used to find 4 limit cycles bifurcating from the center  $O$ .



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