Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 26, No. 4, pp. 877–885, 1996

0362-546X/96 $15.00 + .00

0362-546X(94)00328-9

INJECTIVITY OF POLYNOMIAL LOCAL HOMEOMORPHISMS OF $\mathbf{R}^n$†

ANNA CIMA,‡ ARMENGOL GASULL§ and FRANCESC MAÑOSAS§

‡ Departament de Matemàtica Aplicada II, E.T.S. d'Enginyers Industrials de Terrassa, Universitat Politècnica de Catalunya, Colom 11, 08222 Terrasa, Barcelona, Spain; and

§ Department de Matemàtiques, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

(*Received* 20 *May* 1994; *received for publication* 27 *October* 1994)

Key words and phrases: Real Jacobian conjecture, injectivity, polynomial map, index.

1. INTRODUCTION

Let $F: \mathbf{R}^n \to \mathbf{R}^n$ be a differentiable map and denote by $JF(x)$ its Jacobian matrix at the point $x \in \mathbf{R}^n$. It is well known that if $\det(JF(x)) \neq 0$ for all $x \in \mathbf{R}^n$ then the map F does not need be invertible, although it is a local diffeomorphism at every x. We say that a map F is *polynomial* if its n components are polynomial functions. This paper mainly deal with the problem to ensure when a polynomial map with nonvanishing Jacobian is a global diffeomorphism.

The results that we get are coherent with the following two famous conjectures.

Real Jacobian conjecture [1–3]. Let $F: \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial map such that the determinant of its Jacobian matrix never vanishes. Then F is a diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$.

Jacobian conjecture [4]. Let $F: \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map such that the determinant of its Jacobian matrix is a constant different from zero. Then F is a diffeomorphism of $\mathbf{C}^n$ onto $\mathbf{C}^n$ and F^{-1} is also a polynomial map.

It is well known that to prove both conjectures it suffices to show the injectivity of F (see [1, 5, 6]). Furthermore, from this last assertion it is not difficult to prove that if the Real Jacobian conjecture holds then the Jacobian conjecture also holds: let $F: \mathbf{C}^n \to \mathbf{C}^n$ satisfying the hypotheses of the Jacobian conjecture and let $F^*: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ be the map obtained from F by identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. It is easy to comprove that $\det(JF^*(x)) = |\det JF(z)|^2$, where x and z are the corresponding points of $\mathbf{R}^{2n}$ and $\mathbf{C}^n$ by the above identification. Therefore, the map F^* verifies the hypotheses of the Real Jacobian conjecture. So if the Real Jacobian conjecture holds, then F^* is one to one and, hence, F is one to one. Then the implication is proved. From now on, we will concentrate on the Real Jacobian conjecture.

In [7, 8] the authors give sufficient conditions to ensure that a map satisfying the hypotheses of the Real Jacobian conjecture will be one to one. In the present paper we obtain more general conditions and also we prove injectivity with slightly different conditions. To state the results we need introduce some notation and definitions.

† Supported by a DGCICYT grand number PB93-0860.