PERGAMON

Nonlinear Analysis 35 (1999) 343–354

Nonlinear Analysis

The discrete Markus–Yamabe problem[1]

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Received 21 January 1997; accepted 24 June 1997

Keywords: Asymptotic stability; Markus–Yamabe conjecture; Jacobian conjecture

1. Statement of the problem

Let $F:\mathbf{R}^n\to\mathbf{R}^n$ be a $\mathscr{C}^1$ map and consider the differential system

$$\dot{x}=F(x). \tag{1}$$

Assume that p is a critical point of Eq. (1), i.e., $F(p)=0$. We say that p is a global attractor of the continuous dynamical system induced by Eq. (1) if $\phi(t,x)$ tends to p as t tends to infinity for each $x\in\mathbf{R}^n$, where $\phi(t,x)$ is the solution of Eq. (1) with $\phi(0,x)=x$.

The next conjecture was explicitly stated by Markus and Yamabe (see [15]) in 1960.

MYC (n) (Markus–Yamabe Conjecture). *Let F be a $\mathscr{C}^1$ map from $\mathbf{R}^n$ to itself such that for any $x\in\mathbf{R}^n$, the Jacobian of F at x has all its eigenvalues with negative real part. If $F(p)=0$, then p is a global attractor of $\dot{x}=F(x)$.*

This conjecture was proved for planar polynomial maps in 1988 [16] and for planar $\mathscr{C}^1$ maps in 1993 [9, 11] and in 1994 see [10]. In [1, 3] there are examples of smooth vector fields of $\mathbf{R}^n$, $n\geq 4$ satisfying the hypothesis of the Conjecture which have a periodic orbit and in [4] there is an example of a polynomial vector fields of $\mathbf{R}^n$, $n\geq 3$ satisfying the same hypothesis which has some orbits that scape at infinity. Therefore

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[1] Partially supported by the DGICYT grant number PB96-1153.

PII: S0362-546X(97)00715-3

the conjecture is only true for $n=2$. On the other hand in [6] van den Essen shows that a conjecture weaker than MYC implies the classical Jacobian Conjecture.

Our aim is to discuss a natural translation of MYC to the dynamics of the iterations of maps. Let $F:\mathbf{R}^n \to \mathbf{R}^n$ be a $\mathscr{C}^1$ map and consider the sequence:

$$x^{(m+1)} = F(x^{(m)}), \quad x^{(0)} \in \mathbf{R}^n. \tag{2}$$

Let p be a fixed point of F, i.e., $F(p)=p$. We say that p is a global attractor of the discrete dynamical system (2) if the sequence $x^{(m)}$ tends to p as m tends to infinity for any $x^{(0)} \in \mathbf{R}^n$. The question is the following:

DMYQ(n) (Discrete Markus–Yamabe Question). Let F be a $\mathscr{C}^1$ map from $\mathbf{R}^n$ into itself such that $F(0)=0$ and for any $x \in \mathbf{R}^n$, $JF(x)$ has all its eigenvalues with modulus less than one. Is it true that 0 is a global attractor for the discrete dynamical system generated by F?

The answer of DMYQ(1) is trivially affirmative. Szlenk worked on this problem for $n=2$ and he explained us his result: an example of a rational map $F:\mathbf{R}^2 \to \mathbf{R}^2$ which gives a negative answer to DMYQ(2). Unfortunately Szlenk suddenly died in July 1995. We reproduce his example in Section 5 (see Theorem D) and we prove its properties in the Appendix.

Szlenk's example is neither a polynomial nor a diffeomorphism. However, it can be modified slightly in order to obtain a diffeomorphism which also gives a negative answer to DMYQ(2) (see Theorem E in Section 5).

If we restrict our attention to the family of polynomial maps, we prove that the answer to the DMYQ(2) is affirmative (see Theorem B in Section 2). A negative answer to DMYQ(n) for $n \geq 4$ is given in [7]. Later in [4] there are examples which give a negative answer for $n \geq 3$. So the problem is completely solved.

The paper is organized as follows. In the next section we prove that the answer to the DMYQ is yes for triangular maps defined on $\mathbf{R}^n$ (see Theorem A in Section 2) and also for polynomial maps defined on $\mathbf{R}^2$. In Sections 3 and 4 we relate the problem with the Jacobian Conjecture and with the Markus–Yamabe Conjecture, respectively. Section 5 is devoted to obtaining a diffeomorphism of $\mathbf{R}^2$ which gives a negative answer to DMYQ(2). Lastly in the Appendix we present the Slenzk's example mentioned above. After this paper was written G. Meisters told us that a question similar to DMYQ was already stated in La Salle's book [17] p. 20.

2. The triangular case

We say that F is a triangular map if it takes the form:

$$F(x) = (F_1(x_1), F_2(x_1,x_2), \ldots, F_n(x_1,x_2,\ldots,x_n)).$$

We say that F is linearly triangularizable if there exists a linear change such that it makes F triangular.

The first result which we obtain is the following:

Theorem A. *Let $F:\mathbf{R}^n\to\mathbf{R}^n$ be a $\mathscr{C}^1$ triangular map with all the eigenvalues of $JF(x)$ with modulus less than one at each $x\in\mathbf{R}^n$. Assume that $F(0)=0$. Then 0 is a global attractor for the discrete dynamical system generated by F.*

Proof. We prove the result by induction. The hypothesis of induction is the following:
Consider $F:\mathbf{R}^m\to\mathbf{R}^m$ a triangular $\mathscr{C}^1$ map such that

$$\left|\frac{\partial F_1}{\partial x_1}\right|<1,\quad \left|\frac{\partial F_2}{\partial x_2}\right|<1,\ldots,\left|\frac{\partial F_m}{\partial x_m}\right|<1 \quad\text{and}\quad F(0)=0.$$

Fix $x^{(0)}\in\mathbf{R}^m$ and set $x^{(k+1)}=F(x^{(k)})$. Then there exist $k_0\in\mathbf{N}$ big enough, $M\in\mathbf{R}^+$ and $K\in\mathbf{R}$, $0<K<1$, such that

$$|x_i^{(k+k_0)}|\le MK^k \quad \text{for all } i=1,2,\ldots,m.$$

Obviously if we prove the above induction hypothesis the theorem follows.

The proof for the case $m=1$ is trivial. Assume that it is true for $m=s-1$ and we prove it for $m=s$.

Fix $x^{(0)}\in\mathbf{R}^5$. By considering $\hat{F}(x_1,x_2,\ldots,x_{s-1})\equiv(F_1(x_1),F_2(x_1,x_2),\ldots,F_{s-1}(x_1,x_2,\ldots,x_{s-1}))$ and applying the induction hypothesis we have that there exist k_0, M and K such that

$$|x_i^{(k+k_0)}|\le MK^k \quad \text{for } i=1,2,\ldots,s-1 \text{ and all } K\in\mathbf{N}$$

It is not restrictive to change $x^{(0)}$ by $x^{(k_0)}$ and assume that

$$|x_i^{(k)}|\le MK^k \quad \text{for } i=1,2,\ldots,s-1. \tag{3}$$

Consider

$$\begin{aligned} x_s^{(k)} &= F_s(x^{(k-1)})=F_s(x_1^{(k-1)},x_2^{(k-1)},\ldots,x_s^{(k-1)})\\ &=\int_0^{x_s^{(k-1)}}\frac{\partial F_s}{\partial x_s}(x_1^{(k-1)},x_2^{(k-2)},\ldots,x_{s-1}^{(k-1)},t)\,\mathrm{d}t\\ &\quad+\int_0^{x_{s-1}^{(k-1)}}\frac{\partial F_s}{\partial x_{s-1}}(x_1^{(k-1)},x_2^{(k-2)},\ldots,x_{s-2}^{(k-1)},t,0)\,\mathrm{d}t+\cdots\\ &\quad+\int_0^{x_1^{(k-1)}}\frac{\partial F_s}{\partial x_1}(t,0,\ldots,0)\,\mathrm{d}t. \end{aligned}$$

Then

$$|x_s^{(k)}|\le|x_s^{(k-1)}|+|x_{s-1}^{(k-1)}|+\cdots+|x_1^{(k-1)}|, \tag{4}$$

where we have used that for any $i=1,2,\ldots,s-1$

$$\max\left\{\left|\frac{\partial F_s}{\partial x_i}(y_1,\ldots,y_i,0,\ldots,0)\right|:(y_1,y_2,\ldots,y_{i-1})\in\mathbf{R}^{i-1}\text{ and } -x_i^{(0)}\le y_i\le x_i^{(0)}\right\}$$
$$<1.$$

Now, from Eqs. (3) and (4) we have:

$$|x_s^{(k)}| \le |x_s^{(k-1)}| + (s-1)MK^{k-1}. \tag{5}$$

Denoting $(k-1)M$ by C we can write

$$|x_s^{(k)}| \le |x_s^{(0)}| + C(K^{s-1} + K^{s-2} + \cdots + 1) \le |x_s^{(0)}| + \frac{C}{1-K}.$$

Hence, $\{x_s^{(k)}\}$ is a bounded sequence. Since the same is true for $\{x_i^{(k)}\}$ for $i=1, 2, \ldots, s-1$ there exists $L>0$ such that $|x_i^{(k)}| \le L$ for all $i=1,2,\ldots,s$ and for all $k \in \mathbf{N}$. So, we can define

$$D = \max\left\{ \left| \frac{\partial F_s}{\partial x_s}(y_1, y_2 \ldots, y_s) \right| : |y_i| \le L,\ i=1,2,\ldots,s \right\}$$

and assert that D is strictly less than one. It implies that inequality (4) can now be substituted by

$$|x_s^{(k)}| \le D|x_s^{(k-1)}| + |x_{s-1}^{(k-1)}| + \cdots + |x_1^{(k-1)}|$$

and inequality (5) by

$$|x_s^{(k)}| \le D|x_s^{(k-1)}| + K^{k-1}C.$$

Hence,

$$\begin{aligned}
|x_s^{(k)}| &\le D(D|x_s^{(k-2)}| + K^{k-2}C) + K^{k-1}C \\
&\ \vdots \\
&\le D^k|x_s^{(0)}| + C(K^{k-1} + DK^{k-2} + \cdots + D^{k-1}) \\
&\le D^k|x_s^{(0)}| + kC(\max(K,D))^{k-1} \\
&\le (\max(K,D))^{k-1}(D|x_s^{(0)}| + kC) = (\max(K,D))^k\hat{M} = \hat{M}\hat{K}^k
\end{aligned}$$

for some $\hat{M}$, where $\max(K,D) < \hat{K} < 1$, and the induction is finished. □

The next goal is to show that the answer to DMYQ(2) for polynomial maps is affirmative. First of all notice that in the polynomial case the assumption of the existence of a fixed point can be removed in the hypothesis of DMYQ(n) (this is due to the fact that injective polynomial maps of $\mathbf{R}^n$ are also exhaustive).

Notice that for $n=2$, the results of [9,10] or [11] imply that any map (not necessarily polynomial) satisfying the assumptions of DMYQ(2) has only one fixed point. To see this if F is such a map it suffices to consider the map $F - I$ and conclude that it is injective.

Lemma 1.1. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ (resp. $F : \mathbf{C}^n \to \mathbf{C}^n$) be a polynomial map such that $JF(x)$ has all its eigenvalues with modulus less than one at each $x \in \mathbf{R}^n$ (resp. $x \in \mathbf{C}^n$). Then the characteristic polynomial of $JF(x)$ is independent on x.*

Proof. Let $P_x(\lambda)$ be the characteristic polynomial of $JF(x)$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of $P_x(\lambda)$. Then

$$P_x(\lambda) = \lambda^n - t_1\lambda^{n-1} + \cdots + (-1)^n t_n$$

where

$$t_j = \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, i_1, \ldots, i_j = 1}}^{n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}.$$

Since $|\lambda_i| < 1$ for all $i = 1, 2, \ldots, n$ we have that $|t_j| < k_j$ for all $j = 1, 2, \ldots, n$. On the other hand since the components of F are polynomials and each t_j can be described as the sum of all the minors of order j which have its diagonal on the principal diagonal of $JF(x)$, we conclude that t_j is a polynomial in x for all $j = 1, 2, \ldots, n$. Since the only bounded polynomials are the constants, the result follows. □

Theorem B. *Let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a polynomial map with all the eigenvalues of $JF(x)$ with modulus less than one at each $x \in \mathbf{R}^2$. Then there exists a unique fixed point of F which is a global attractor for the discrete dynamical system generated by F.*

Proof. Set $F = (P, Q)$. From Lemma 1.1 we know that

$$t_1 = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \quad \text{and} \quad t_2 = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}$$

are constants. Consider $G = F - (t_1/2)I$ and set $G = (\bar{P}, \bar{Q})$. Then,

$$\bar{t}_1 = \frac{\partial \bar{P}}{\partial x} + \frac{\partial \bar{Q}}{\partial y} = 0 \quad \text{and} \quad \bar{t}_2 = \frac{\partial \bar{P}}{\partial x}\frac{\partial \bar{Q}}{\partial y} - \frac{\partial \bar{P}}{\partial y}\frac{\partial \bar{Q}}{\partial x} = t_2 - \frac{t_1^2}{4}.$$

Since $\bar{t}_1 = 0$, there exists a polynomial $H(x, y)$ such that

$$\bar{P}(x, y) = -\frac{\partial H}{\partial y} \quad \text{and} \quad \bar{Q}(x, y) = \frac{\partial H}{\partial x}.$$

Then the hessian of H is $\bar{t}_2$. If $\bar{t}_2 \neq 0$, by applying the result of [5] we know that up a complex affine transformation,

$$H(u, v) = \sqrt{-\bar{t}_2}\,uv + h(u)$$

where h is a polynomial in one variable u.

Assume $\bar{t}_2 < 0$. Then, $H(u, v) \in \mathbf{R}$ and from the proof of [5] we see that the affine transformation can be taken with real coefficients. Through this transformation we have that $F(x, y) = (P(x, y), Q(x, y))$ can be written as $(\tilde{P}(u, v), \tilde{Q}(u, v))$ where

$$\tilde{P}(u, v) = k_1 u + k_2 \quad \text{and} \quad \tilde{Q}(u, v) = k_3 v + p(u) + k_4$$

with p a polynomial in one variable. With these coordinates $(\tilde{P}, \tilde{Q})$ has an unique fixed point $(\bar{u}, \bar{v})$. By doing the translation which sends $(\bar{u}, \bar{v})$ to $(0, 0)$ and applying Theorem 1, the result follows.

Now assume that $\bar{t}_2>0$. From the classical result of Jörgens (see [14]), we know that $H(x,y)$ is quadratic. It implies that $F=(P,Q)$ is a polynomial map of degree one. For this type of maps the result can be easily proved.

If $\bar{t}_2=0$ from the proof of Dillen it follows that, up to an affine transformation, $H(u,v)=kv+h(u)$, $k\in\mathbf{R}$, and the result follows as above. □

3. The Jacobian Conjecture and the fixed point conjecture

In this section we restrict our attention to polynomial maps. Let $F:\mathbf{R}^n\to\mathbf{R}^n$ be a polynomial map such that $JF(x)$ has all its eigenvalues with modulus less than one. In Section 1 we ask for the existence of a fixed point which is a global attractor. Now we formulate a weaker problem as follows:

FPC (Fixed Point Conjecture). *Let $F:\mathbf{R}^n\to\mathbf{R}^n$ be a polynomial map such that $JF(x)$ has all its eigenvalues with modulus less than one at each $x\in\mathbf{R}^n$. Then F has a unique fixed point.*

Considering the real and the imaginary part of the components of a complex map $F:\mathbf{C}^n\to\mathbf{C}^n$ and using standard arguments of linear algebra it is easy to see that this conjecture can be formulated in the following equivalent form:

FPC (Fixed Point Conjecture). *Let $F:\mathbf{C}^n\to\mathbf{C}^n$ be a polynomial map such that $JF(x)$ has all its eigenvalues with modulus less than one at each $x\in\mathbf{C}^n$. Then F has a unique fixed point.*

Theorem C shows that this problem is equivalent to the celebrated Jacobian Conjecture, which can be established as follows.

JC (Jacobian Conjecture). *Let $F:\mathbf{C}^n\to\mathbf{C}^n$ be a polynomial map with $\det JF(x)\in\mathbf{C}^*=\mathbf{C}\backslash\{0\}$ at each $x\in\mathbf{C}^n$. Then F is invertible.*

Theorem C.

JC *is equivalent to* FPC.

Proof. Assume that the JC holds and let F satisfy the hypothesis of FPC for some n. Consider $G(x)=F(x)-x$. Then the eigenvalues of $JG(x)$ are the eigenvalues of $JF(x)$ minus one. Hence by using Lemma 1.1 we have that $\det JG(x)$ is constant; from the hypothesis on F we know that this constant cannot be zero. So, G is invertible and there exists a unique x_0 zero of G, which is the unique fixed point of $F(x)$.

Now assume that JC fails for some m. From the Reduction Theorem (see [2]) it means that there exists $n\in\mathbb{N}$ and $G:\mathbf{C}^n\to\mathbf{C}^n$ noninvertible such that

$$G(x)=x+H(x)$$

with $JH(x)$ a nilpotent matrix at each $x\in\mathbf{C}^n$. Now set $g(x)=\frac{1}{2}G(x)$ and let $y,z\in\mathbf{C}^n$, $y\neq z$ with $g(y)=g(z)=p$. Denoting by $h(x)$ the expression $x+p-g(x)$, we have that $h(y)=y$ and $h(z)=z$. On the other hand, since $JH(x)$ is a nilpotent matrix at each $x\in \mathbf{C}^n$, all its eigenvalues are zero. From the definition of $h(x)$ we obtain

$$Jh(x)=I - Jg(x)=I - (\tfrac{1}{2}I + \tfrac{1}{2}JH(x))=\tfrac{1}{2}I - \tfrac{1}{2}JH(x)$$

which implies that all the eigenvalues of $Jh(x)$ are $\frac{1}{2}$ at each point $x\in \mathbf{C}^n$. Hence, $h(x)$ satisfies the hypothesis of FPC and it has two different fixed points. □

Remember that the results of [9, 10] or [11] imply that FPC for $\mathbb{R}^2$ is true. From this fact, and the proof of Theorem C it can be deduced that the JC is true for some special subcases and it cannot be deduced that it is true for $n=2$ as can be thought at a first look.

To end this section we give a conjecture which implies the JC. Firstly we explain the way we arrive at the conjecture. Notice that the DMYQ has affirmative answer for triangularizable maps. So we thought that the next step could be to consider it for maps that take the form $F(x)=(f(x),g(x_n))$ with $g:\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{R}^n\to\mathbb{R}^{n-1}$. We had some problems to solve DMYQ for this case and we decided to consider the easiest case in which F is polynomial and all the eigenvalues of $JF(x)$ are zero. That is, we tried to solve DMYQ for polynomial maps of the form $F(x)=(f(x),0)$ where $JF(x)$ is nilpotent. At this point we wonder if all polynomial maps with nilpotent Jacobian can be reduced to that case:

NC (Nilpotent Conjecture). *Let $F:\mathbb{C}^n\to\mathbb{C}^n$ be a polynomial map such that JF is nilpotent. Then there exists a linear change of coordinates A such that $AFA^{-1}(x)=(f(x),0)$ where f is a polynomial map of $\mathbb{C}^n$ into $\mathbb{C}^{n-1}$.*

We have not been able to solve the above conjecture. van den Essen has independently arrived at the next result.

Proposition 3.1.
NC *implies* JC.

Proof. It is known that it suffices to prove JC for maps F such that $JF=I+N$ with N nilpotent and homogeneous of degree 3. Therefore, assuming that NC holds, we will prove by induction over n next statement: Any map F from $\mathbf{C}^n$ into $\mathbf{C}^n$ such that $JF=I+N$ with N nilpotent is injective.

For $n=1$ the above statement is trivial. Suppose that it is true in dimension n and let $F:\mathbf{C}^{n+1}\to\mathbf{C}^{n+1}$ be such that $JF=I+N$ with N nilpotent. Assuming that NC holds: the map F, by means a linear change of coordinates, can be written as

$$F(x_1,\dots,x_{n+1})=(x_1+P_1(x_1,\dots,x_{n+1}),x_2+P_2(x_1,\dots,x_{n+1}),\dots,x_{n+1}).$$

To show that this map is injective it suffices to show that for any $a\in\mathbf{C}$ the map

$$F_a(x_1,\dots,x_n)=(x_1+P_1(x_1,\dots,x_n,a),x_2 +P_2(x_1,\dots,x_n,a),\dots,x_n+P_n(x_1,\dots,x_n,a))$$

is injective. But clearly F_a satisfies the induction hypothesis and we are done. □

After the first version of this paper was finished van den Essen and Hubbers have given a counterexample of NC, for all $n \geq 2$ and F of degree greater or equal than 4; see [8]. Observe that following the proof of Proposition 3.1 we can prove next result, which shows that NC is yet an open approach to JC.

Proposition 3.2. NC *for polynomial maps of degree less or equal than* 3 *implies* JC.

4. Relation with the Markus–Yamabe conjecture

In this section we relate the MYC(n) with DMYQ(n). Consider the differential system

$$\dot{x} = F(x) \tag{6}$$

and let $\phi(t,x)$ be the solution of Eq. (6) with $\phi(0,x) = x$.

Assume that $\phi(T,x)$ is defined for some $T > 0$ and for some $x \in \mathbf{R}^n$. Then we can consider the discrete dynamical system given by the flow at time T:

$$\phi(T,\cdot) : V \to \mathbf{R}^n$$

where V is a neighbourhood of x.

Lemma 4.1. *Let F be a $\mathscr{C}^1$ map from $\mathbf{R}^n$ to itself and let $\phi(t,x)$ be the solution of $\dot{x} = F(x)$ with $\phi(0,x) = x$. Then the following hold*:

(i) *If for all $x \in \mathbf{R}^n$ $JF(x)$ has all its eigenvalues with negative real part, then given U a bounded open subset of R^n there is $T > 0$ such that for all $t \in (0,T)$ the Jacobian of $\phi(t,\ ) : U \to \mathbf{R}^n$ has all its eigenvalues with modulus less than one at any point of U.*

(ii) *Assume that $(\mathrm{d}/\mathrm{d}x)\phi(t,x)$ has all its eigenvalues with modulus less than one for $t \in (0, t_x]$. Then $JF(x)$ has all its eigenvalues with nonpositive real parts.*

Proof. Since $\phi(t,x)$ is the solution of $\dot{x} = F(x)$ with $\phi(0,x) = x$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,x) = F(\phi(t,x)).$$

Taking the derivatives with respect to x and evaluating in $t = 0$ we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\phi(t,x))\right)\Bigg|_{t=0} = JF(x).$$

This last equality can be rewritten as

$$\lim_{t\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\phi(t,x)) - I}{t} = JF(x), \tag{7}$$

where I means the scalar identity matrix. From the above relation it is clear that for each $y \in \mathscr{C}l(U)$ ($\mathscr{C}l(U)$ means the topological closure of U) there exists a neighbourhood of y, U_y and a positive real number T_y such that for each $z \in U_y$ the matrix

$(\mathrm{d}/\mathrm{d}x)\phi(t,x)(z)$ has all its eigenvalues with real part less than one (resp. greater than one), for each $t\in(0,T_y)$ (resp. $t\in(-T_y,0)$). Since ϕ is continuous and $\phi(0,x)=x$, there exist W_y neighbourhood of y and a positive real number γ_y with $\gamma_y<T_y$ such that $\phi(t,z)\in U_y$ for all $z\in W_y$ and for all $t\in(-\gamma_y,\gamma_y)$.

Let $z\in W_y$ and $t\in(0,\gamma_y)$. Since $\phi(-t,\phi(t,x))=x$, by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(-t,x)(\phi(t,x))\frac{\mathrm{d}}{\mathrm{d}x}\phi(t,x)(x)=I.$$

Let $\lambda=a+b\mathrm{i}$ be an eigenvalue of $(\mathrm{d}/\mathrm{d}x)\phi(t,x)(z)$. Then $\lambda^{-1}=(a-b\mathrm{i})/(a^2+b^2)$ is an eigenvalue of $(\mathrm{d}/\mathrm{d}x)\phi(-t,x)(\phi(t,z))$. Since $z,\phi(t,z)\in U_y$ and $t\in(0,T_y)$ we obtain $\operatorname{Re}\lambda<1$ and $\operatorname{Re}\lambda^{-1}>1$. This clearly implies that $|\lambda|=a^2+b^2<1$.

We have seen that for each $y\in\mathscr{C}l(U)$ there exists W_y neighbourhood of y and a positive number γ_y such that $(\mathrm{d}/\mathrm{d}x)\phi(t,x)(z)$ has its eigenvalues with modulus less than one for each $z\in W_y$ and for all $t\in(0,\gamma_y)$. Let $W_{y_1},\dots,W_{y_k}$ be a finite cover of $\mathscr{C}l(U)$ and $T=\min\{\gamma_{y_1},\dots,\gamma_{y_k}\}$. Clearly $T>0$ and (i) follows.

To prove (ii) it is enough to consider equality (7). □

The set of the values of t for which part (1) of the above lemma holds cannot be extended to $\mathbf{R}^+$, because of the known following lemma and the counterexamples with periodic orbits; see [3].

Lemma 4.2 (Hartman [13]). *Assume that system* (1) *has a periodic orbit of period T. Then the flow at time T has an eigenvalue equal to one.*

If the answer to the DMYQ for flows (which are special cases of diffeomorphisms) was affirmative, then we could conclude that system (6) has a global attractor under the hypotheses that for any $x\in\mathbf{R}^n$, $\phi(t,x)$ is defined for some $t>0$ and $\phi(t,\cdot)$ has eigenvalues with modulus less than one. Observe that these last hypotheses are stronger than the Markus–Yamabe ones.

5. The answer to DMYQ(n) for diffeomorphisms

In this section we obtain a rational diffeomorphism of $\mathbf{R}^n$ ($n\geq 2$) which satisfies the hypothesis of DMYQ and has a periodic orbit. This example is a modification of Szlenk's example which is described in the following theorem that will be proved in the Appendix.

Theorem D (Szlenk). *Let* $F:\mathbf{R}^2\to\mathbf{R}^2$ *be defined by*

$$F(x,y)=\left(-\frac{ky^3}{1+x^2+y^2},\frac{kx^3}{1+x^2+y^2}\right)\quad \textit{where } k\in\left(1,\frac{2}{\sqrt{3}}\right).$$

The map F satisfies the following properties:

(1) *Set* $p=(x,y)\in\mathbf{R}^2$ *and let* λ *be an eigenvalue of* $JF(p)$. *If* $xy=0$ *then* $\lambda=0$. *Otherwise* $\lambda\notin\mathbf{R}$ *and* $|\lambda|<\sqrt{3}k/2$.

(2) $F^4(1/\sqrt{k-1},0)=(1/\sqrt{k-1},0)$.
(3) *F is injective.*

Let $G_a:\mathbf{R}^n\to\mathbf{R}^n$ be defined by

$$G_a(x_1,x_2,x_3,\dots,x_n)=\left(-\frac{kx_2^3}{1+x_1^2+x_2^2}-ax_1,\frac{kx_1^3}{1+x_1^2+x_2^2}-ax_2,cx_3,\dots,cx_n\right)$$

where $|c|<1$ and $k\in(1,2/\sqrt{3})$.

Theorem E. *For a small enough, the map G_a is a global diffeomorphism from $\mathbf{R}^n$ into itself which satisfies the following properties:*
(1) *For all $x\in\mathbf{R}^n$ and for all λ eigenvalue of $(DG_a)(x)$, $|\lambda|<1$.*
(2) *$G_a(0)=0$ and there exists $p\in\mathbf{R}^n$, $p\neq 0$ which satisfies $G_a^4(p)=p$.*

Proof. Clearly we can assume that $n=2$. Note that $G_a=F-aI$ where F is as in Theorem D.

Since JF has at each point eigenvalues with modulus less than $(\sqrt{3}/2)k$, it follows that for a small enough JG_a has at each point eigenvalues with modulus less than one. Note that since JF has all eigenvalues different from a, JG_a has at any point eigenvalues different from zero. So G_a is a local diffeomorphism at every point of $\mathbf{R}^2$. In order to see that G_a is a global diffeomorphism we will apply Hadamard's Theorem [12] which asserts that a smooth map from $\mathbf{R}^n$ to itself is a global diffeomorphism if and only if it is a local diffeomorphism at each point of $\mathbf{R}^n$ and it is proper (i.e. the preimage of a compact set is also a compact set). To see that G_a is proper it suffices to show that

$$\lim_{|x,y|\to\infty}|G_a(x,y)|=\infty.$$

Easy computations show that

$$|G_a(x,y)|^2=\frac{k^2r^6(\cos^6\theta+\sin^6\theta)+a^2r^2(1+r^2)^2-kar^4\sin 4\theta(1+r^2)/2}{(1+r^2)^2}$$

where r and θ denote the usual polar coordinates associated to (x,y). Using that $\cos^6\theta+\sin^6\theta\geq 1/4, \sin 4\theta\leq 1$ and $1+r^2<2r^2$ for r big enough we get

$$\begin{aligned}|G_a(x,y)|^2&\geq\frac{k^2r^6}{4(1+r^2)^2}+a^2r^2-\frac{kar^4}{2(1+r^2)}\\&\geq\frac{k^2r^6}{16r^4}+a^2r^2-\frac{kar^2}{2}\\&=(k/4-a)^2r^2\end{aligned}$$

and hence $\lim_{r\to\infty}G_a(x,y)=\infty$. Therefore G_a is a global diffeomorphism for all $a\neq 0$.

Set $p=(1/\sqrt{k-1},0)$. Easy computations show that

$$JF^4(p)=\begin{pmatrix}(3-(2/k))^4 & 0\\ 0 & 0\end{pmatrix}.$$

Therefore the eigenvalues of $JF^4(p)$ are of modulus different from one and so the periodic orbit is hyperbolic. Hence the periodic orbit remains by small perturbations of the map. Hence for a small enough G_a has also a four periodic orbit. This ends the proof of the theorem.

Acknowledgements

We want to thank Arno van den Essen for interesting discussions. His help has been very important in several parts of this paper, specially in the proof of Theorems B and E.

Appendix (W. Szlenk)

This appendix is devoted to proving Theorem D which is due to W. Szlenk. This theorem gives an example of a global smooth homeomorphism (not diffeomorphism) which gives a negative answer to DMYQ(2).

Proof of Theorem D. We only prove (1). The proof of (2) and (3) are simple comprobations. By easy computations we get

$$\det JF(x,y)=\frac{3k^2x^2y^2(x^2+y^2+3)}{(1+x^2+y^2)^3}\quad\text{and}\quad trJF(x,y)=\frac{-2kxy(x^2-y^2)}{(1+x^2+y^2)^2},$$

where $tr(\cdot)$ denotes the trace operator. Using polar coordinates we obtain

$$\begin{aligned}\det JF(r,\theta)&=\frac{3k^2\sin^2\theta\cos^2\theta r^4(r^2+3)}{(1+r^2)^3}\\ &=\frac{3k^2\sin^2 2\theta(r^6+3r^4)}{4(r^6+3r^4+3r^2+1)}<\frac{3k^2}{4}\le 1.\end{aligned}$$

On the other hand we obtain that $\Delta(x,y)$, the discriminant of the characteristic polynomial of $JF(x,y)$, satisfies

$$\begin{aligned}\Delta(x,y)&=tr^2\,JF(x,y)-4\det JF(x,y)\\ &=-4k^2x^2y^2\frac{2x^4+8x^2y^2+2y^4+12x^2+12y^2+9}{(1+x^2+y^2)^4}\le 0.\end{aligned}$$

If $x \cdot y = 0$ then $\det JF(x,y) = 0$ and $tr\, JF(x,y) = 0$ and hence $\lambda = 0$. Otherwise $\Delta(x,y) < 0$ and hence the eigenvalues of $JF(x,y)$ are not real. Therefore, in this case

$$|\lambda| = \sqrt{\det JF(x,y)} < \sqrt{\frac{3k^2}{4}} = \frac{k\sqrt{3}}{2}.$$

Thus (1) follows. □

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