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A note on LaSalle's problems

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Abstract. In LaSalle's book "The Stability of Dynamical Systems", the author gives four conditions which imply that the origin of a discrete dynamical system defined on $\mathbb{R}$ is a global attractor, and proposes to study the natural extensions of these conditions in $\mathbb{R}^n$. Although some partial results are obtained in previous papers, as far as we know, the problem is not completely settled. In this work we first study the four conditions and prove that just one of them implies that the origin is a global attractor in $\mathbb{R}^n$ for polynomial maps. Then we note that two of these conditions have a natural extension to ordinary differential equations. One of them gives rise to the well known Markus–Yamabe assumptions. We study the other condition and we prove that it does not imply that the origin is a global attractor.

1. Introduction and statement of the results. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map and consider the dynamics of iterations of T:

$$x_{k+1} = T(x_k). \tag{1}$$

Assume that 0 is a fixed point of T. We say that it is a *global attractor* for (1) if the sequence x_k tends to 0 as k tends to infinity for any $x_0 \in \mathbb{R}^n$.

Let $A = (a_{ij})$ be a real $n \times n$ matrix. We denote by $\sigma(A)$ the spectrum of A, i.e., the set of eigenvalues of A, and we define $|A| = (|a_{ij}|)$. We also denote by $T'(x) = (\partial T_i(x)/\partial x_j)$ the Jacobian matrix of T at $x \in \mathbb{R}^n$. When $T(0) = 0$, we can write $T(x)$ in the form $T(x) = A(x)x$, where $A(x)$ is an $n \times n$ matrix function. Note that this $A(x)$ is not unique.

LaSalle [11] gives some possible generalizations of the sufficient conditions to have a global attractor for $n = 1$. Concretely, the conditions are the following:

(A_1) $\quad |\lambda| < 1$ for each $\lambda \in \sigma(A(x))$ and for all $x \in \mathbb{R}^n$,

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