Journal of Difference Equations and Applications,
Vol. 10, No. 6, May 2004, pp. 549–559

On Periodic Rational Difference Equations of Order k

ANNA CIMA*, ARMENGOL GASULL† and FRANCESC MAÑOSAS‡

Deptartament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici Cc, 08193 Bellaterra, Barcelona, Spain

(Received 5 September 2003; In final form 24 January 2004)

This paper is devoted to the study of which rational difference equations of order k, with non negative coefficients, are periodic. Our main result is that for $k \leq 5$ and for $k = 7, 9, 11$ only the well known periodic difference equations and their natural extensions appear.

Keywords: Rational difference equations; Non negative coefficients; Periodic difference equations; Periodic rational recurrences

INTRODUCTION

In this paper, we consider the k-order rational difference equation

$$u_{n+k} = \frac{A_1 u_n + A_2 u_{n+1} + \cdots + A_k u_{n+k-1} + A_0}{B_1 u_n + B_2 u_{n+1} + \cdots + B_k u_{n+k-1} + B_0}, \tag{1}$$

with initial condition $(u_1, u_2, \ldots, u_k) \in (0, \infty)^k$, and $A_i \geq 0$, $B_i \geq 0$, $\sum_{i=0}^{k} A_i > 0$, $\sum_{i=0}^{k} B_i > 0$. We assume that $A_1 + B_1 \neq 0$ to ensure that Eq. (1) is of order k.

We study the characterization of the difference equations (1) such that, for all $n \geq 1$ and for any initial condition, $p \in \mathbb{N}$ exists such that $u_{n+p} = u_n$. If p is the minimum natural number with this property, we will say that Eq. (1) is p-periodic. The determination of first order p-periodic rational recurrences ($k = 1$) is well known, see for instance Ref. [4, pp. 453–454]. The problem that we consider has been proposed in Ref. [5] for $k = 2$. See also Ref. [2]. As we will see, the fact that we are only considering positive coefficients which makes the problem more interesting. In fact, without this restriction, for $k = 1$ there are already p-periodic difference equations for all $p \in \mathbb{N}$, (see again Ref. [4]).

First of all, we give a result that restricts the difference equations of type (1) that can be periodic for some $p \in \mathbb{N}$, (see Theorem 2.4). As a consequence of this result, of the study of the periodic points associated to the dynamical system generated by Eq. (1) and of the results of Ref. [1], see Theorem 2.5, we can characterize all the difference equations (1) which are p-periodic for $k = 1, 2, 3, 4, 5, 7, 9$ and 11, (see Theorems 3.1 and 4.1). As far as we know our results for $k \geq 2$ are new.

*Corresponding author. E-mail: cima@mat.uab.es
†E-mail: gasull@mat.uab.es
‡E-mail: manyosas@mat.uab.es

Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190410001667977