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DYNAMICS OF SOME RATIONAL DISCRETE DYNAMICAL SYSTEMS VIA INVARIANTS

ANNA CIMA* and ARMENGOL GASULL†
Dept. de Matemàtiques, Edifici C,
Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Spain
**cima@mat.uab.es*
†gasull@mat.uab.es

VÍCTOR MAÑOSA
Dept. de Matemàtica Aplicada III,
Control, Dynamics and Applications Group (CoDALab),
Universitat Politècnica de Catalunya
Colom 1, 08222 Terrassa, Spain
victor.manosa@upc.edu

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We consider several discrete dynamical systems for which some invariants can be found. Our study includes complex Möbius transformations as well as the third-order Lyness recurrence.

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1. Introduction

Consider a discrete dynamical system (DDS for short)

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \tag{1}$$

where $F : \mathcal{U} \subset \mathbb{K}^m \to \mathbb{K}^m$, with $\mathcal{U}$ being an open subset, and where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

We will say that H is a *nonautonomous invariant* for (1) if H is a function defined in $\mathcal{V}$, an open and dense subset of $\mathcal{U}$, valued in $\mathbb{K}$ and satisfying:

$$H(F(\mathbf{x})) = \xi H(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{V},$$

for some nonzero $\xi \in \mathbb{K}$, which will be called the *multiplier* of H.

We introduce the above definition motivated from a similar concept used when studying nonautonomous (or time-dependent) first integrals for ordinary differential equations, see for instance [Goriely, 2001, Chap. 2].

Notice that when a nonautonomous invariant H has multiplier $\xi = 1$ then we get H as an *invariant*, also called *first integral* for (1), i.e.

$$H(F(\mathbf{x})) = H(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{V}. \tag{2}$$

The goal of the paper is to use first integrals and nonautonomous invariants to study the dynamics of several DDS coming from some rational difference equations. These invariants are obtained by using an extension of the method presented in [Gasull & Mañosa, 2002], which is summarized in the Appendix. Observe that if H is a nonautonomous invariant for F, then $\xi^{-n}H(F^n(\mathbf{x}_0)) = H(\mathbf{x}_0)$, for all $n \in \mathbb{N}$. In particular, this property can be used to determine some properties of the limiting behavior of a DDS in terms of the initial condition.

In Sec. 2, we consider Möbius (or linear fractional) transformations. Although the dynamics associated with these transformations is common